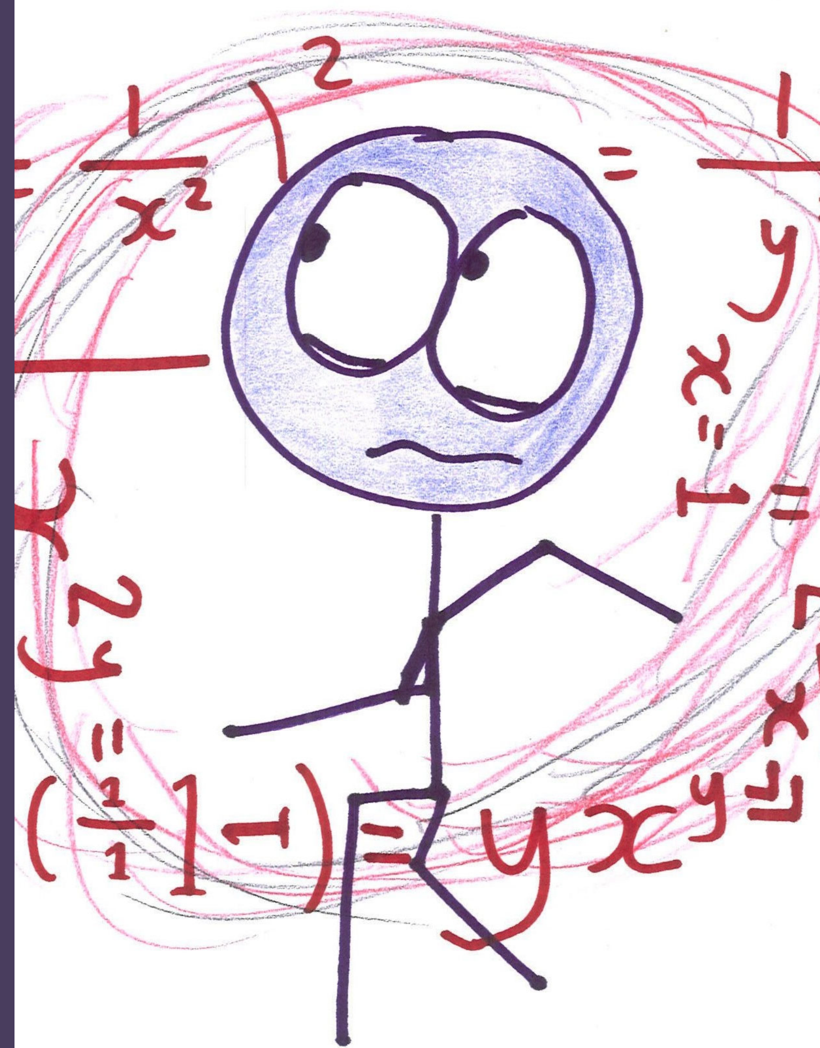
What Does Algebra Look Like to Students?

Ben Orlin

Math with Bad Drawings



Example: Can we understand the formula for a correlation coefficient?

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

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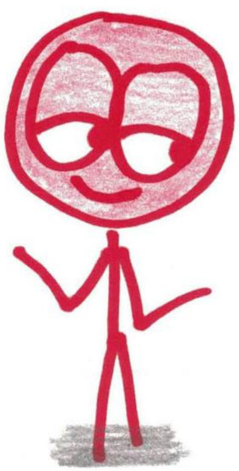
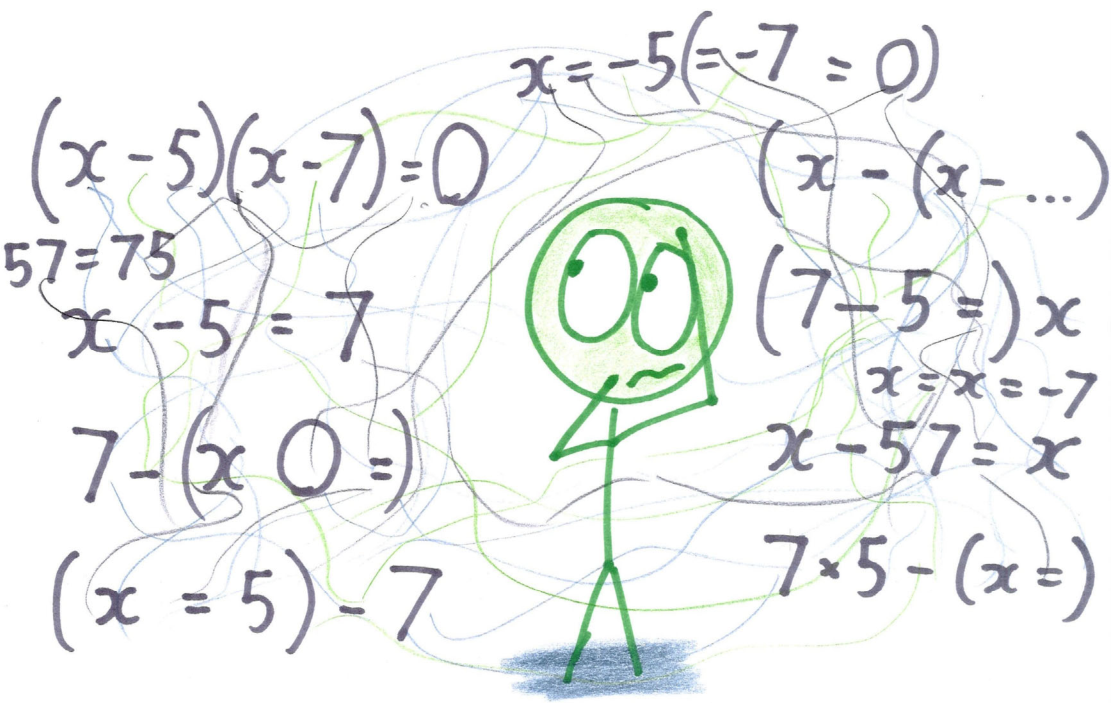
"Ah yes, of course."



"Um… yeah, looks right."

"Kinda losing me, bro."

"PLEASE. NO."



Algebraic notation is language.

Experts have reading strategies.

Novices may not.

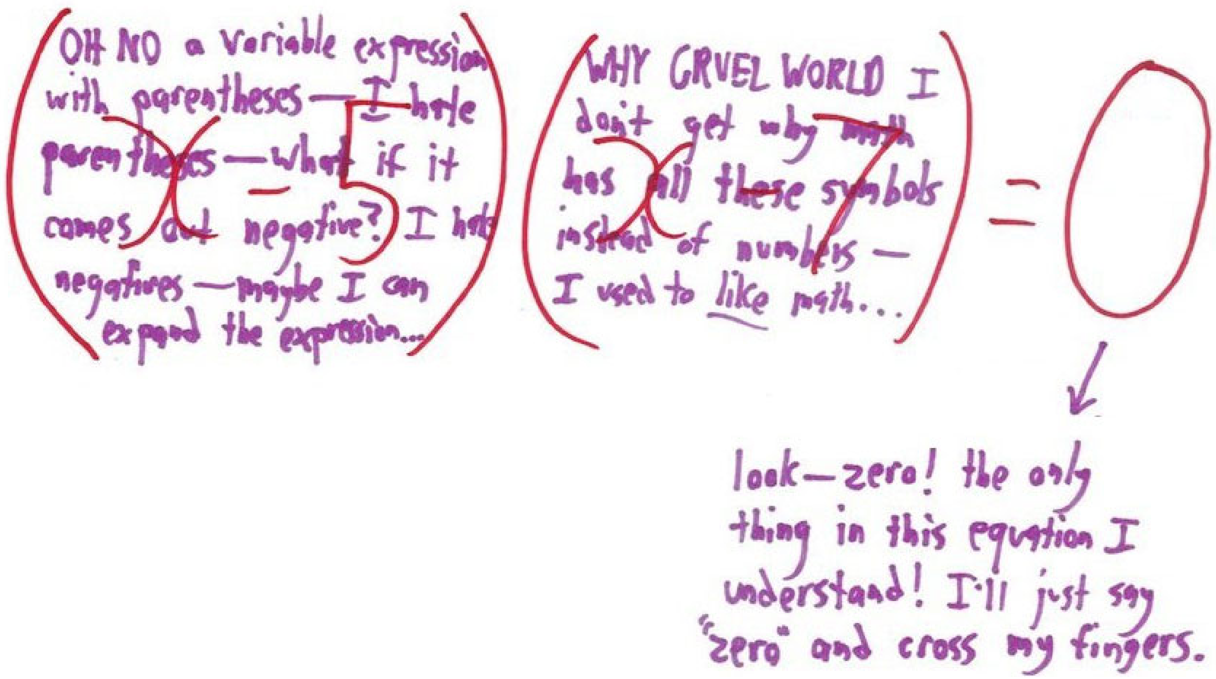
A challenge for teachers: Identify and unpack these reading strategies.

What do novices and experts see?

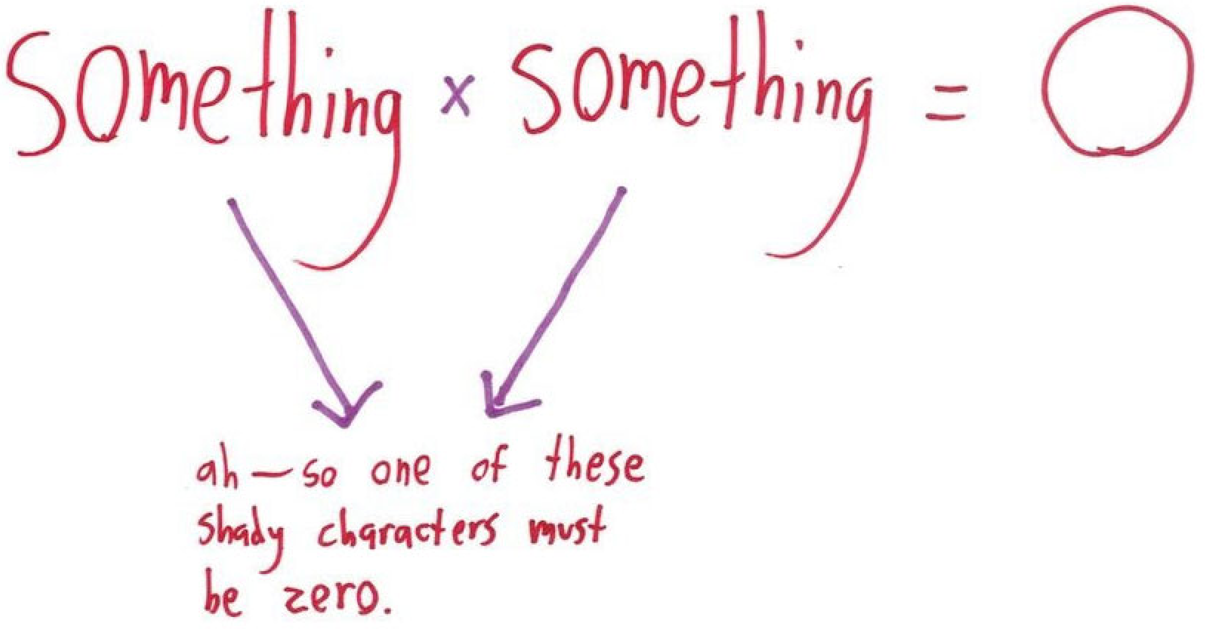
$$(x-5)(x-7)=0$$

Solve for x.

What A Novice Sees



What An Expert Sees

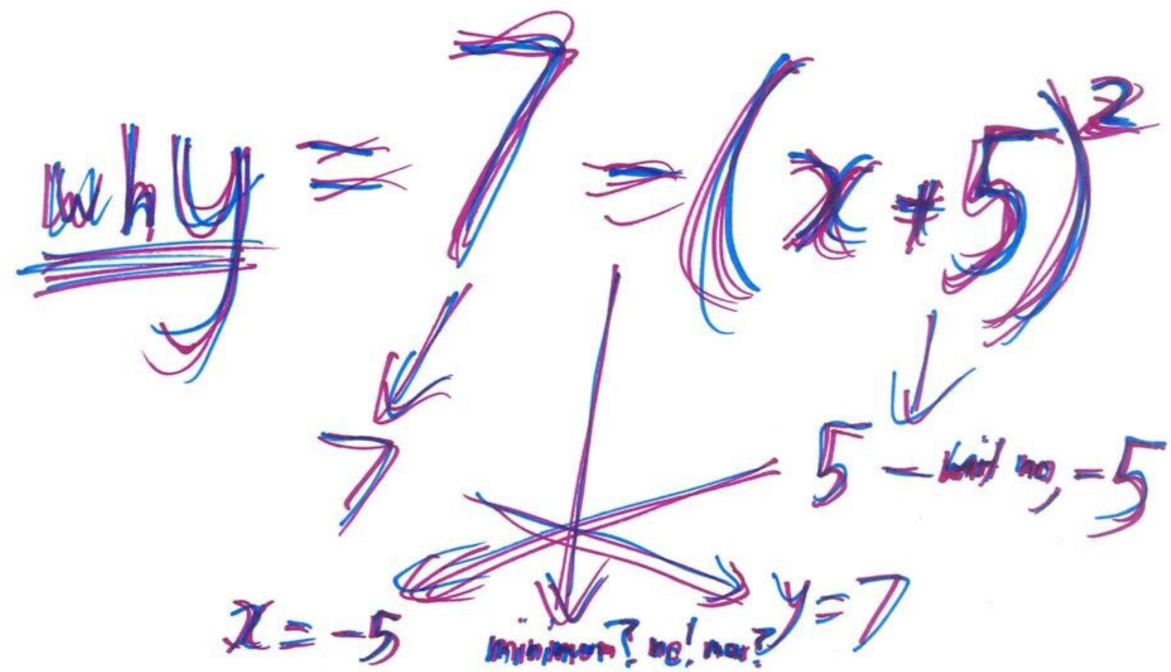


What do novices and experts see?

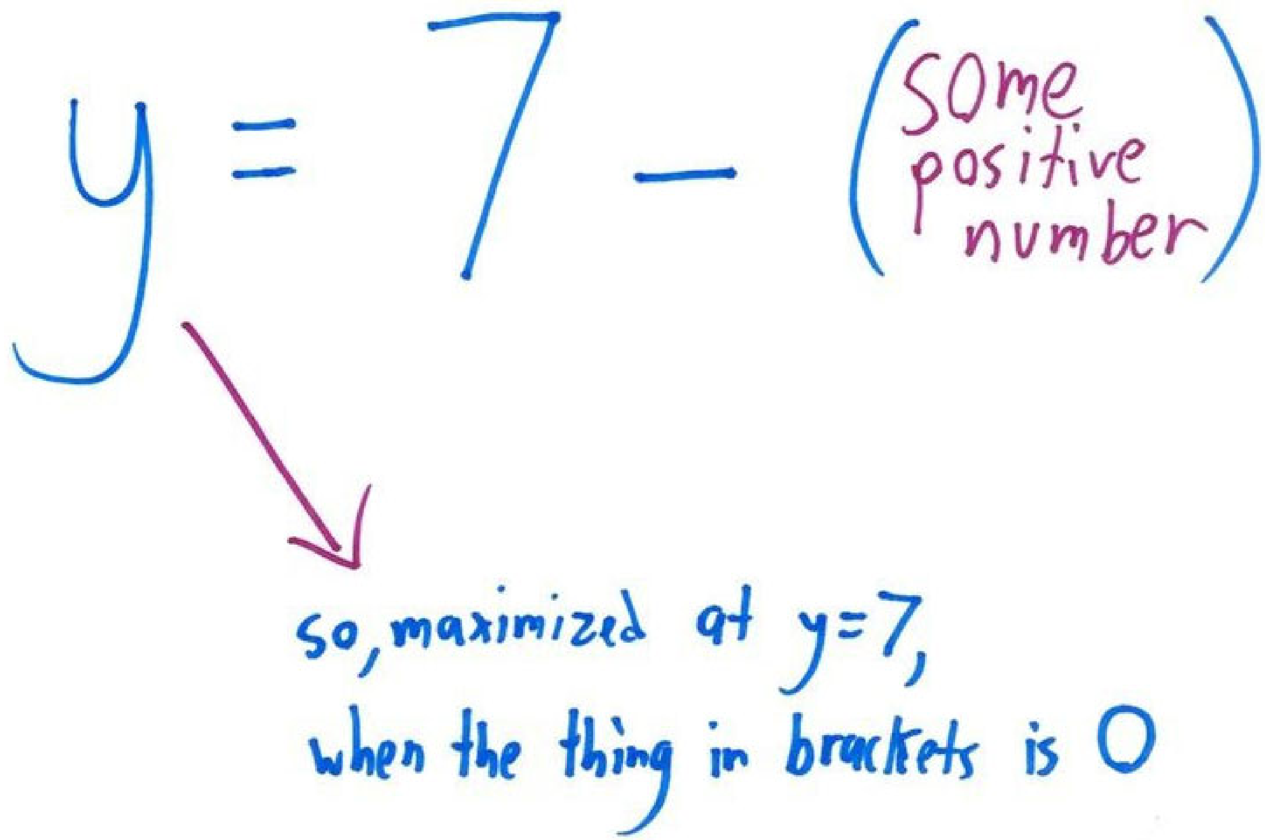
$$y = 7 - (x+5)^2.$$

Find the graph's vertex.

What A Novice Sees



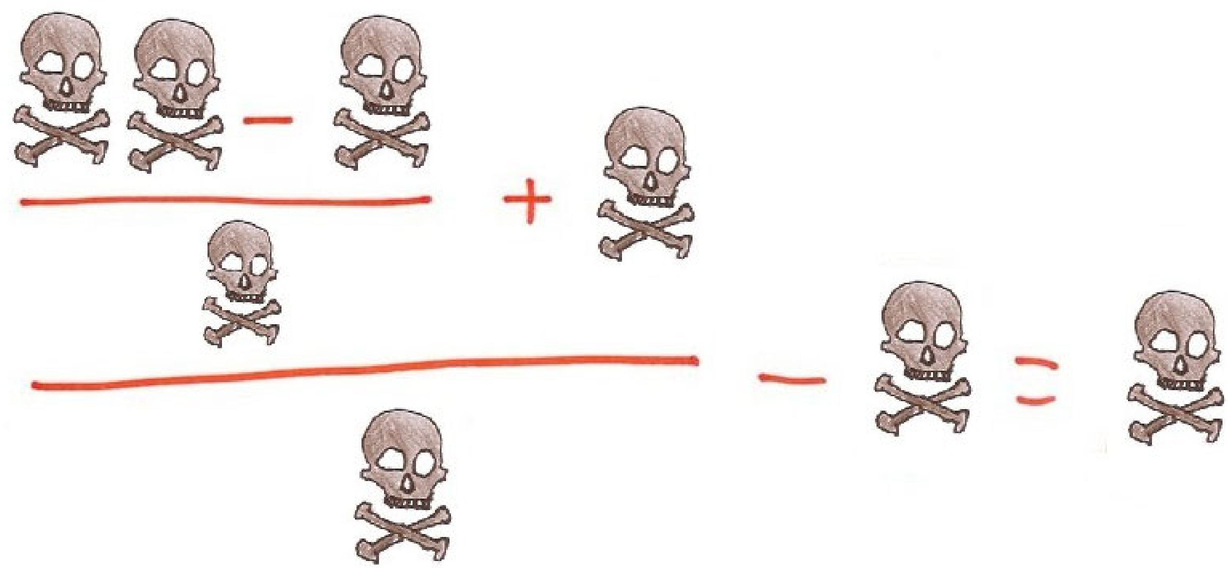
What An Expert Sees



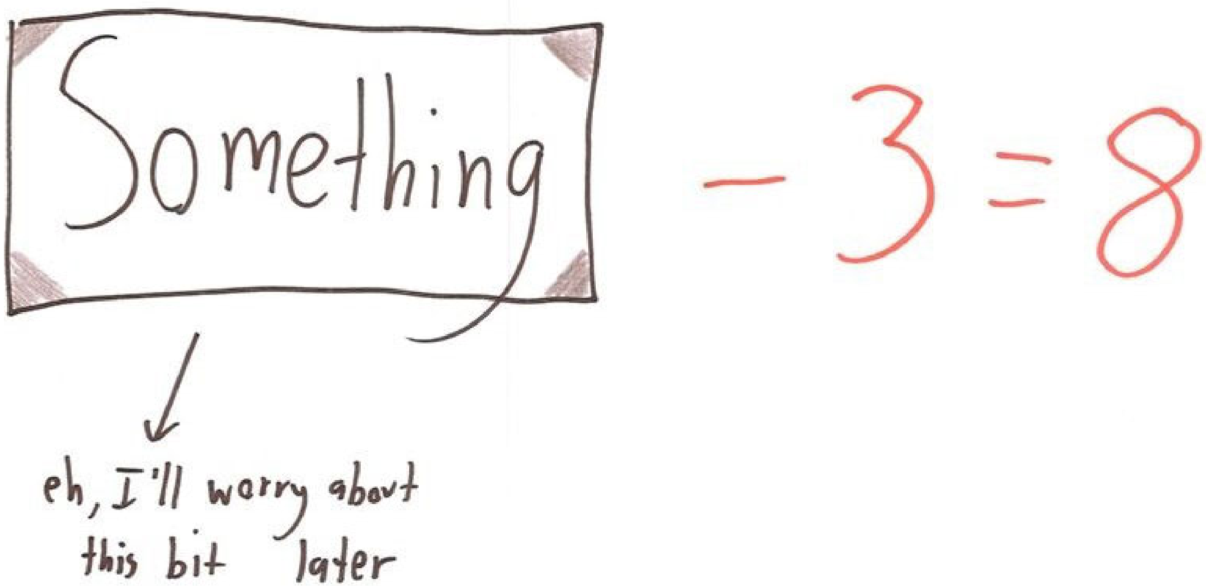
What do novices and experts see?

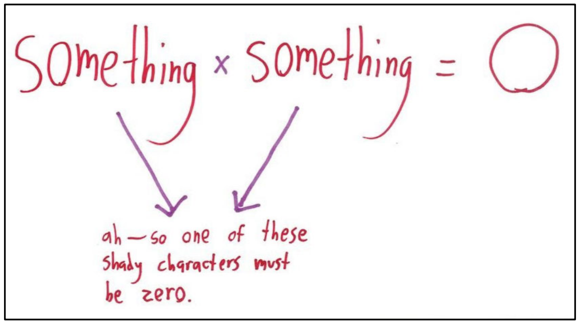
$$\frac{\frac{7x-1}{2}+4}{7}-3=8$$

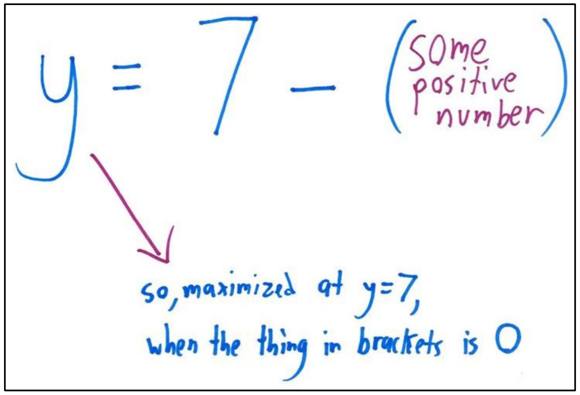
What a Novice Sees

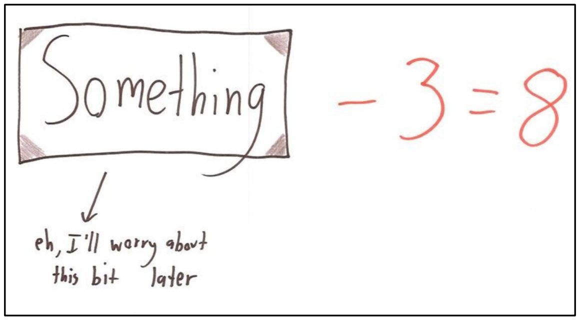


What an Expert Sees









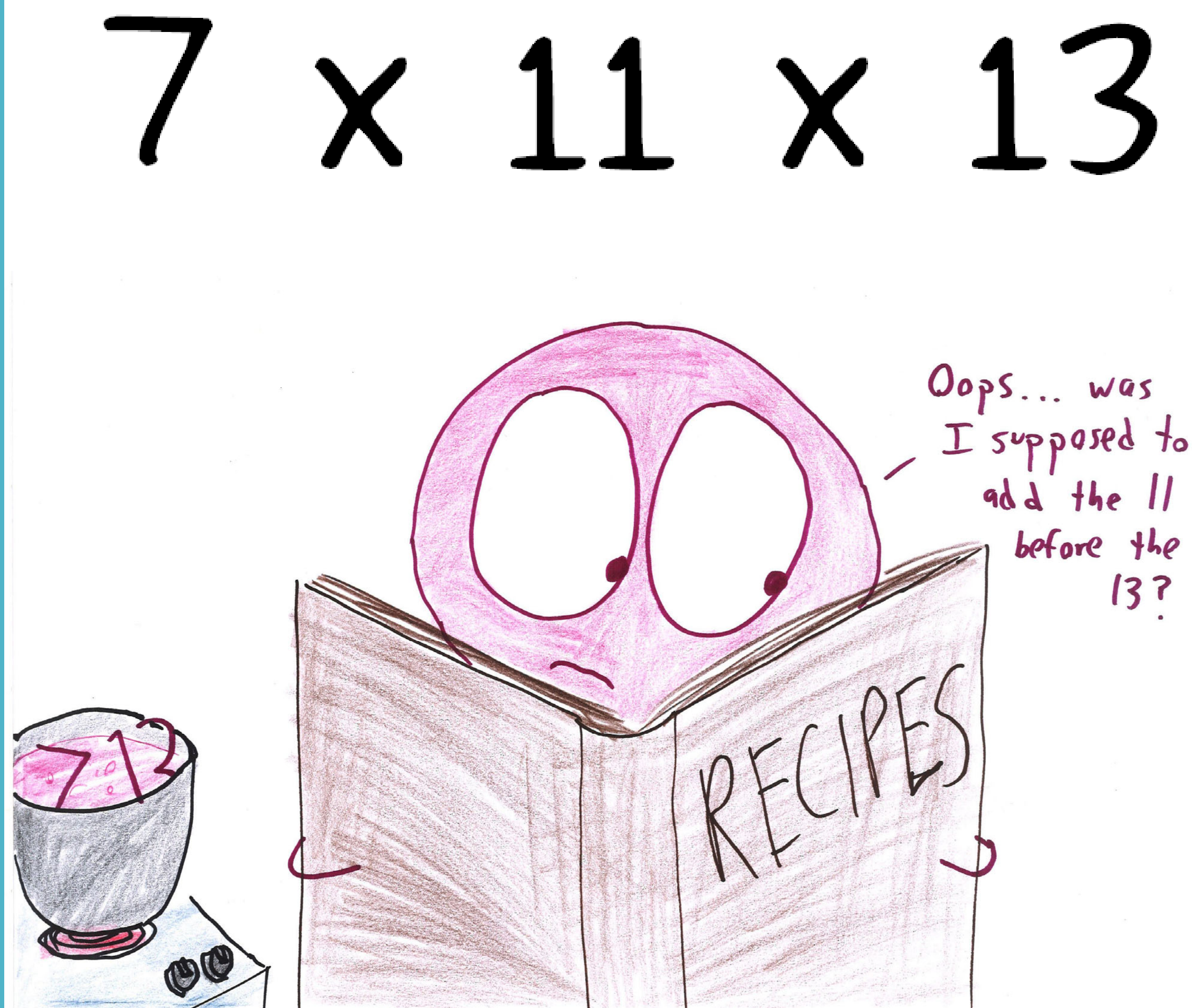
Each of these is an example of what a psychologist would call **chunking**.

You break a large whole into a few meaningful pieces.

What do novices and experts see?

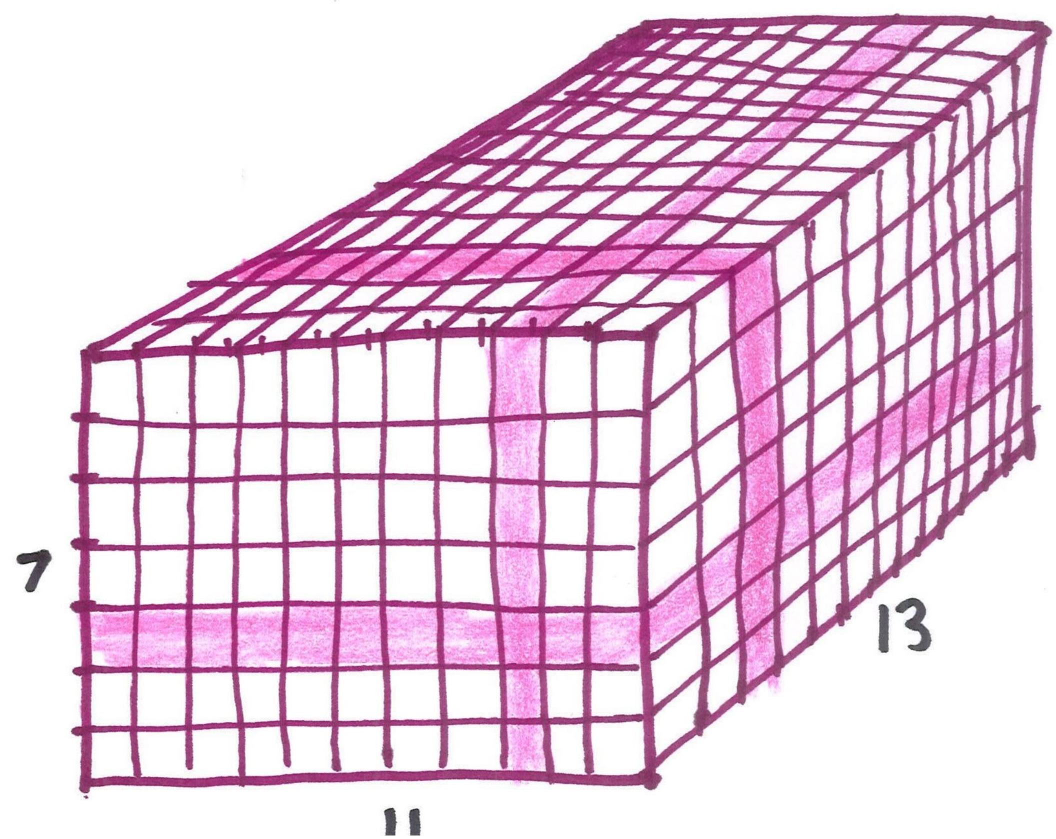
$7 \times 11 \times 13$

What a novice sees



What an expert sees

$7 \times 11 \times 13$

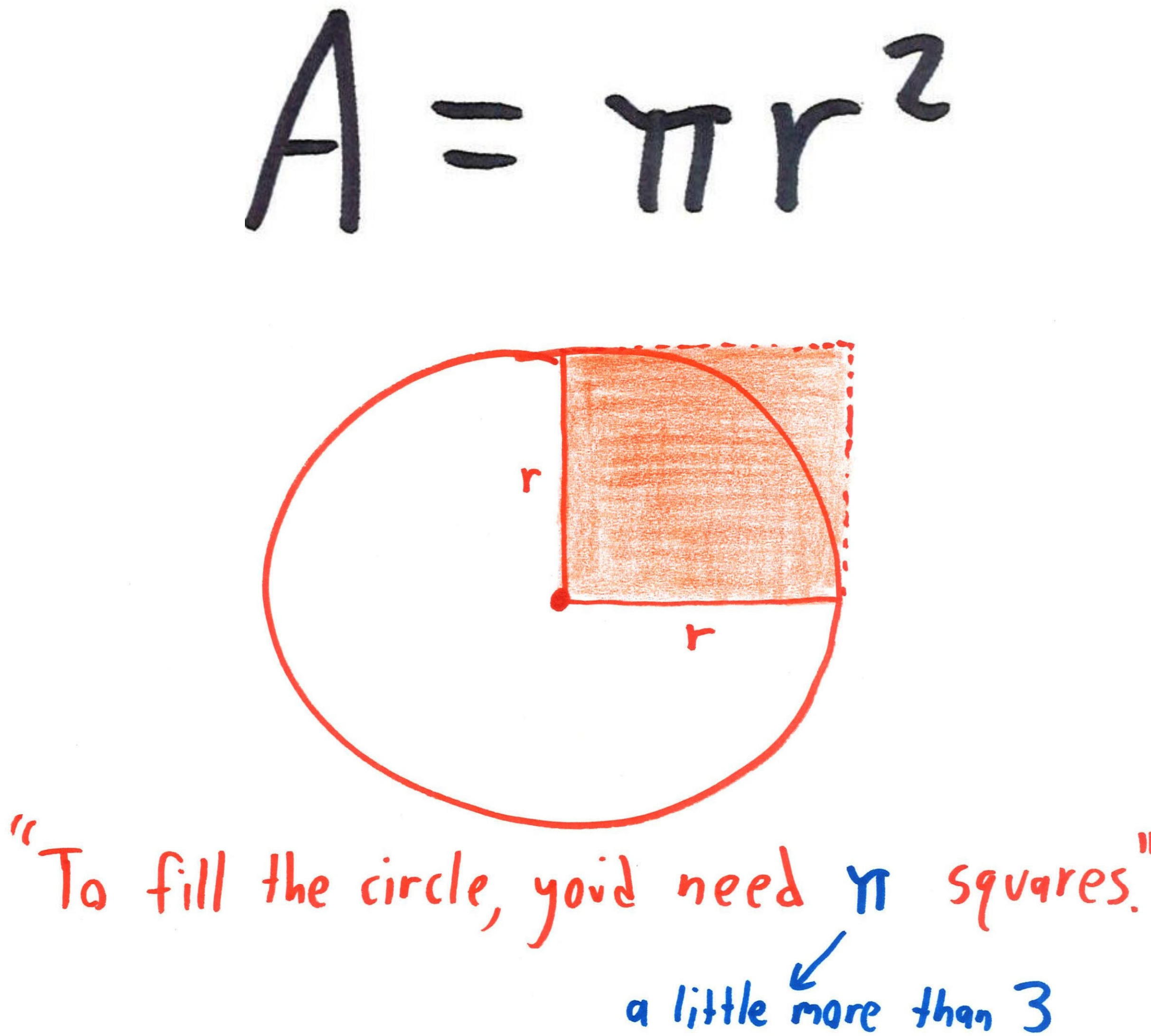


What do novices and experts see?

$$A = \pi r^2$$

What a Novice Sees

What an Expert Sees

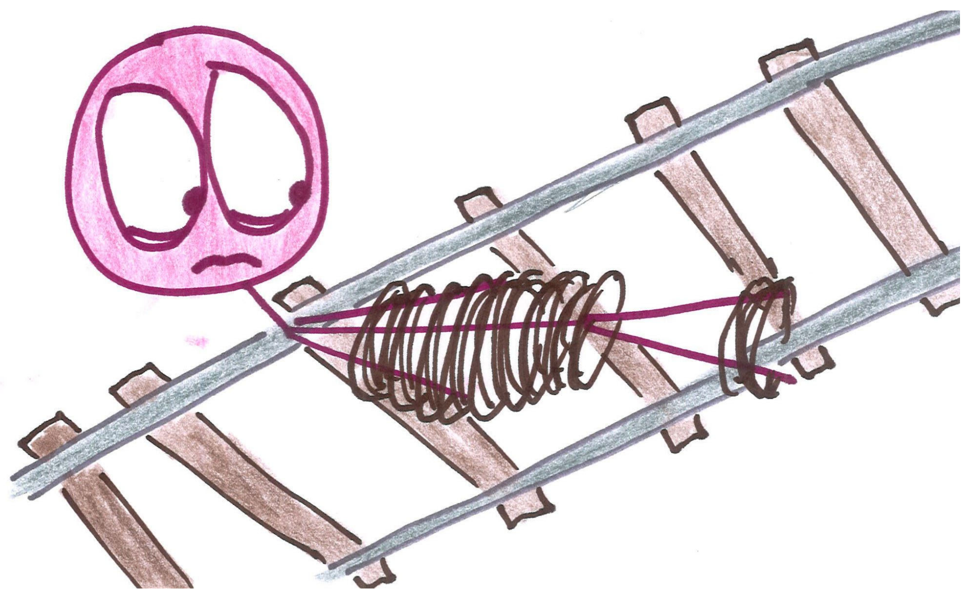


What do novices and experts see?

$$x^2-1 = (x-1)(x+1)$$

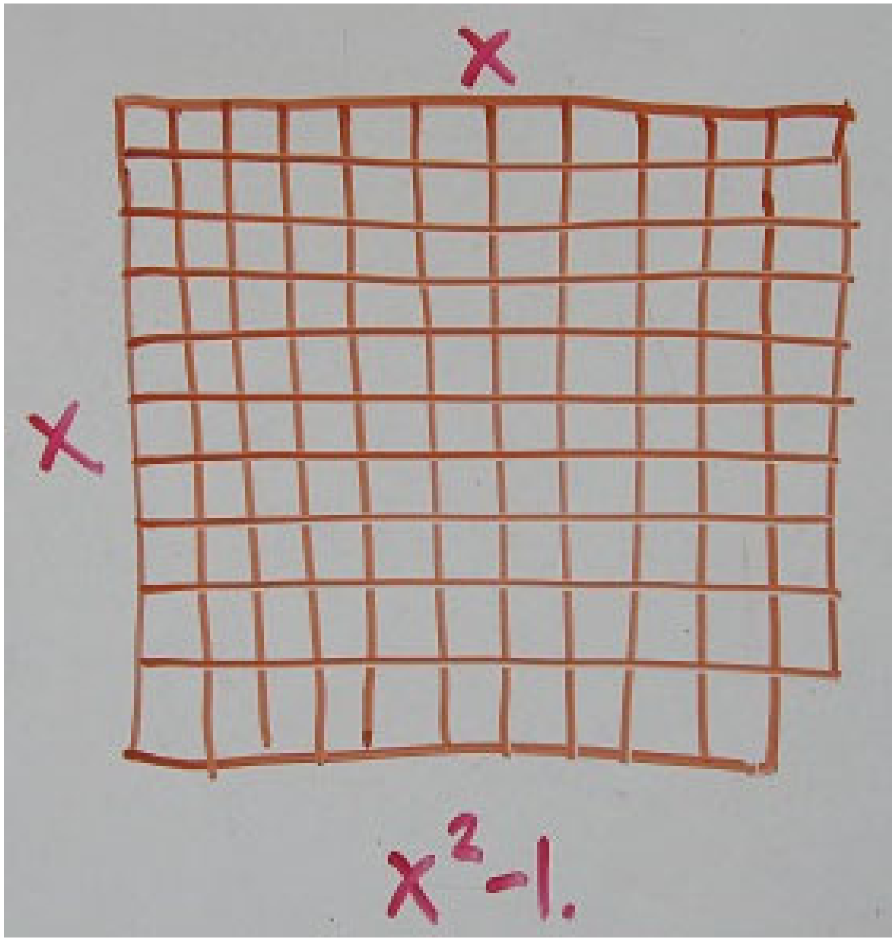
What a novice sees

$$x^2 - 1 = (x-1)(x+1)$$



What an Expert Sees

$$x^2 - 1 = (x-1)(x+1)$$



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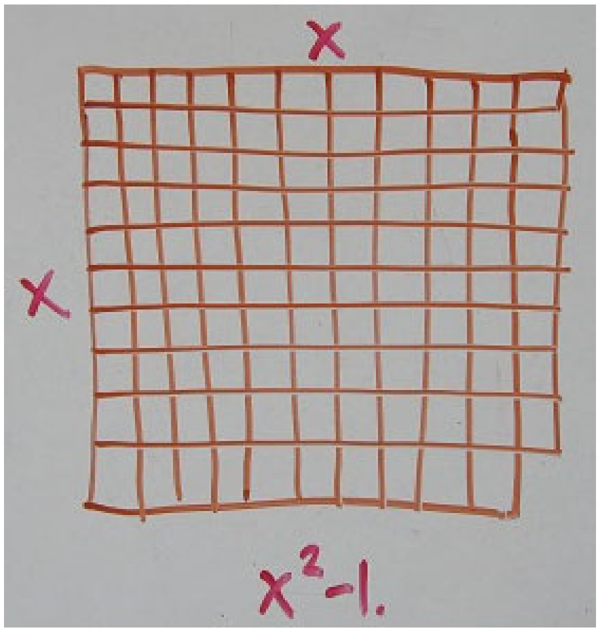
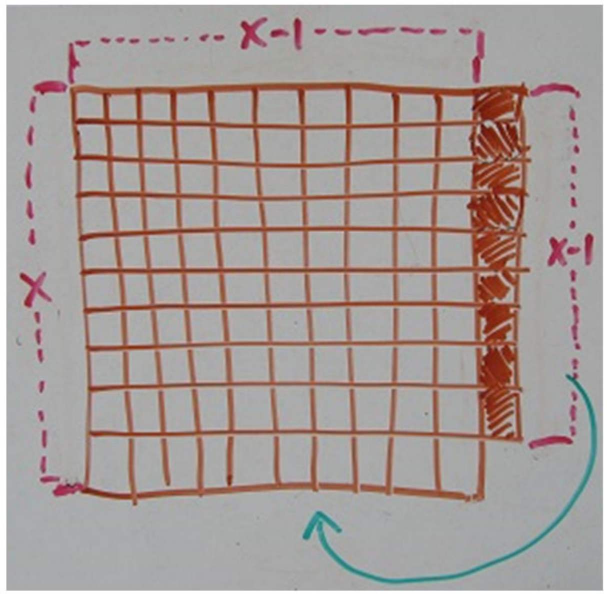
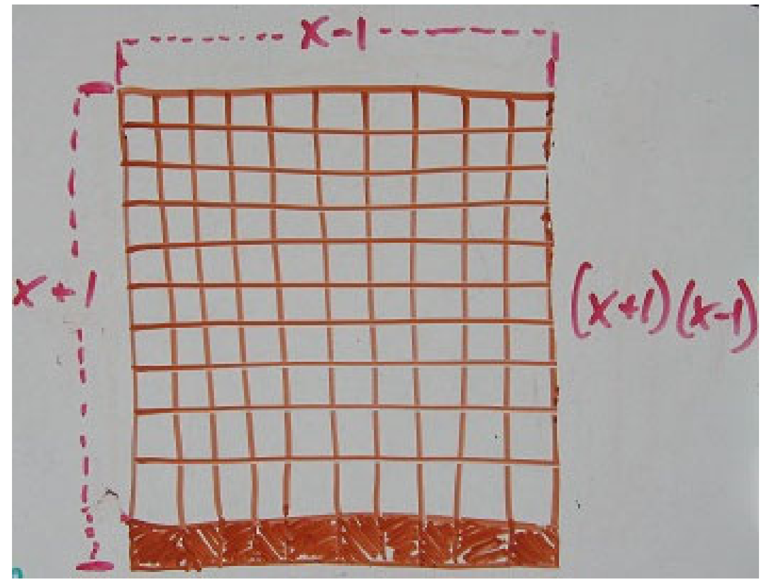
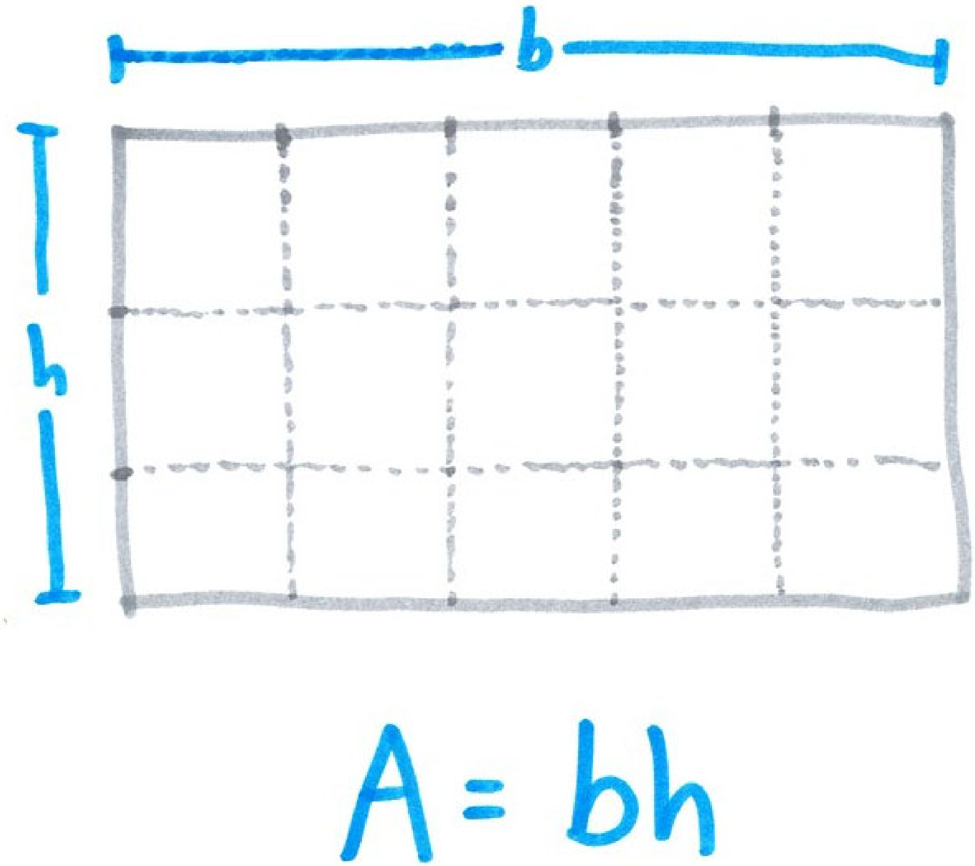








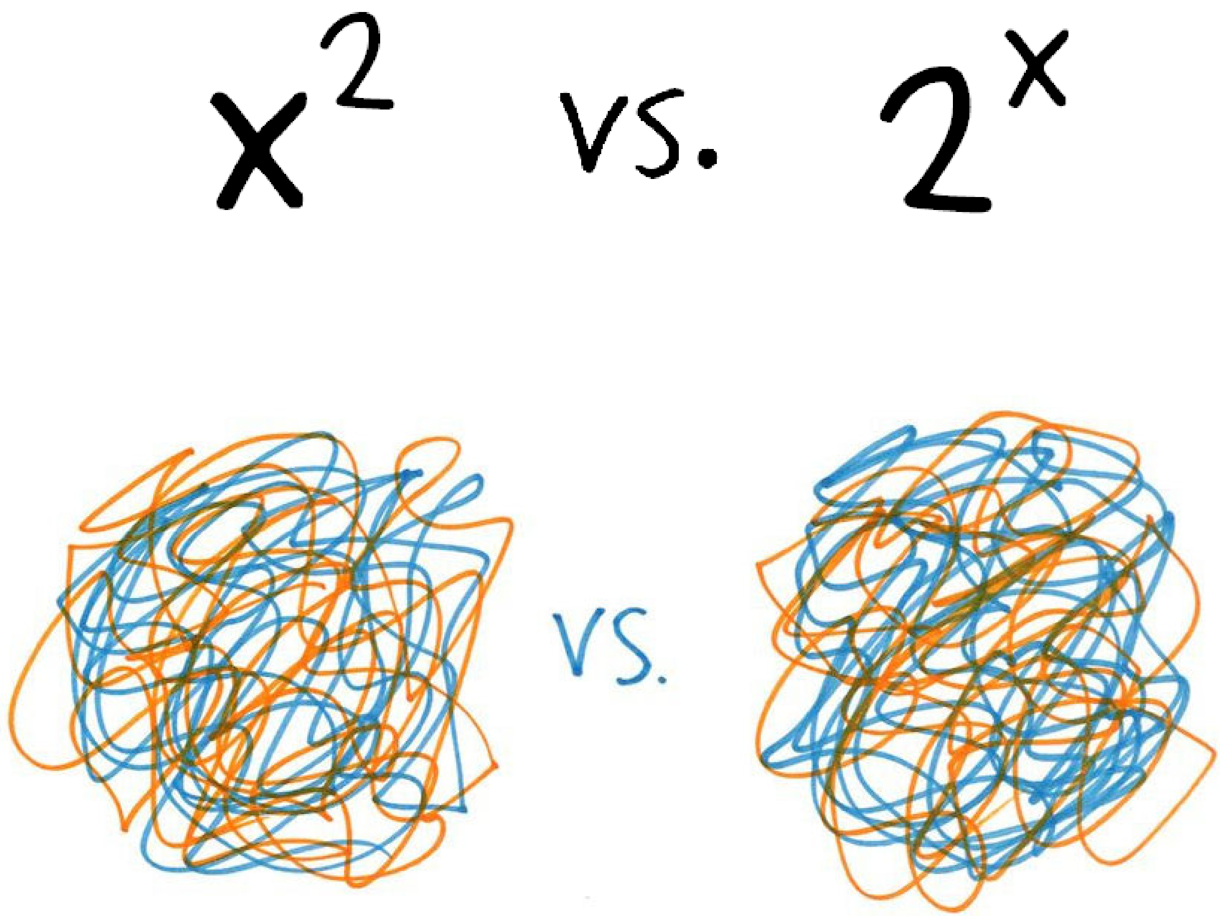
I find **area models** really powerful for making sense of algebra.

But I struggle to get my students on board – at least, at first.

What do novices and experts see?

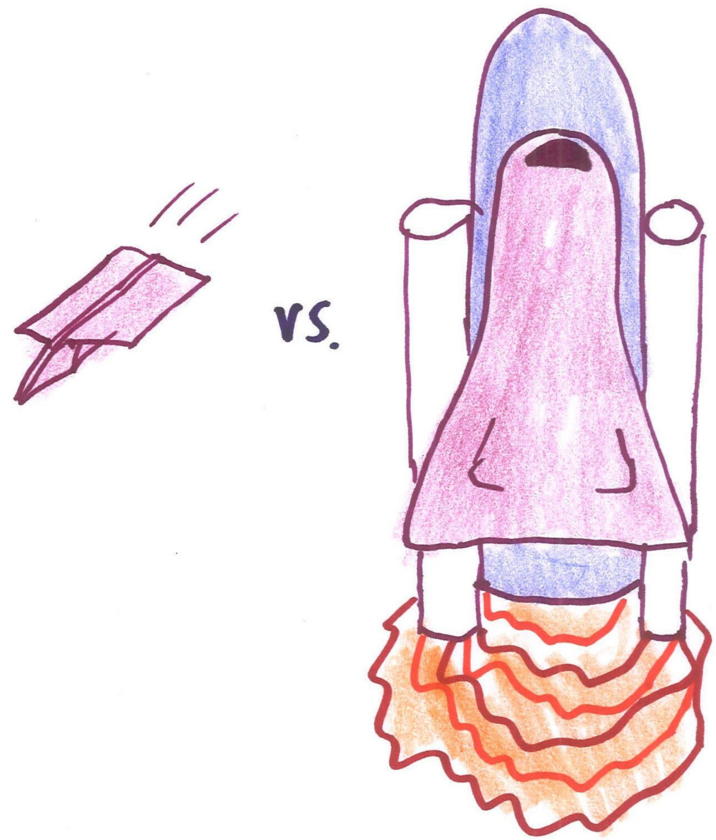
x^2 vs. 2^x

What a Novice Sees

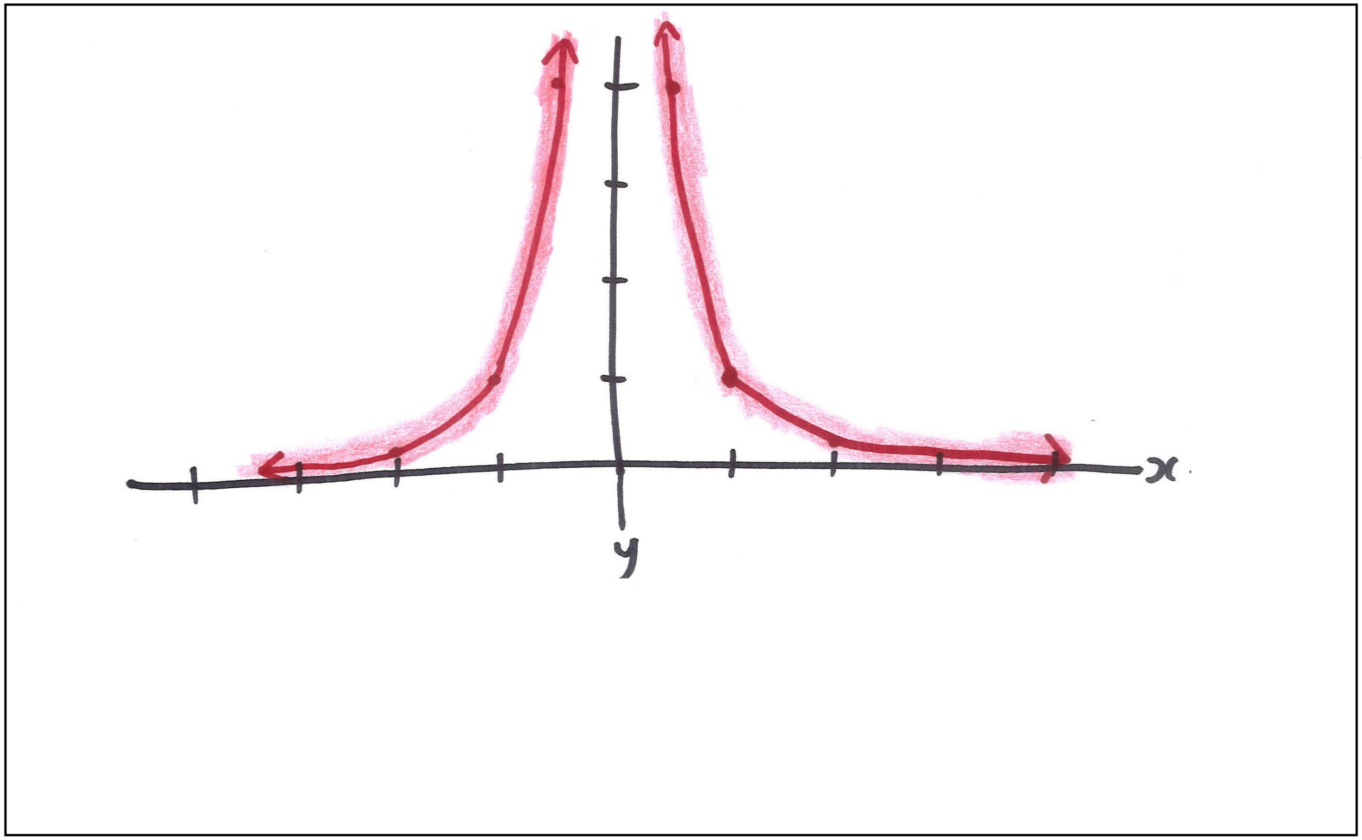


What an Expert Sees

x^2 vs. 2^x



$$y = \frac{1}{x^2}$$



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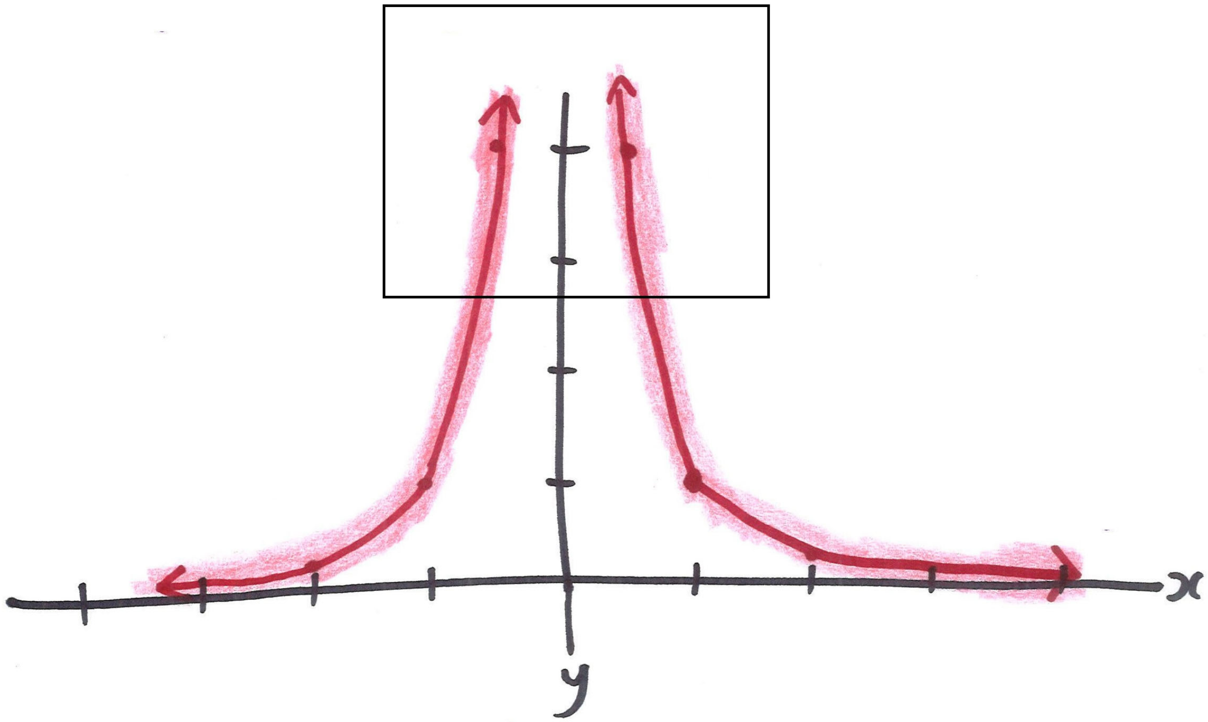
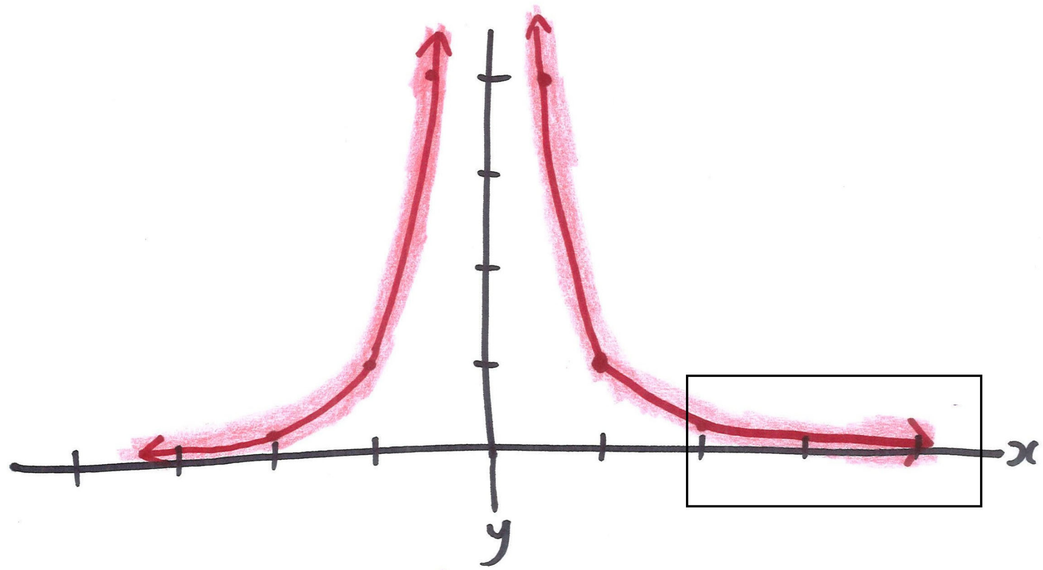


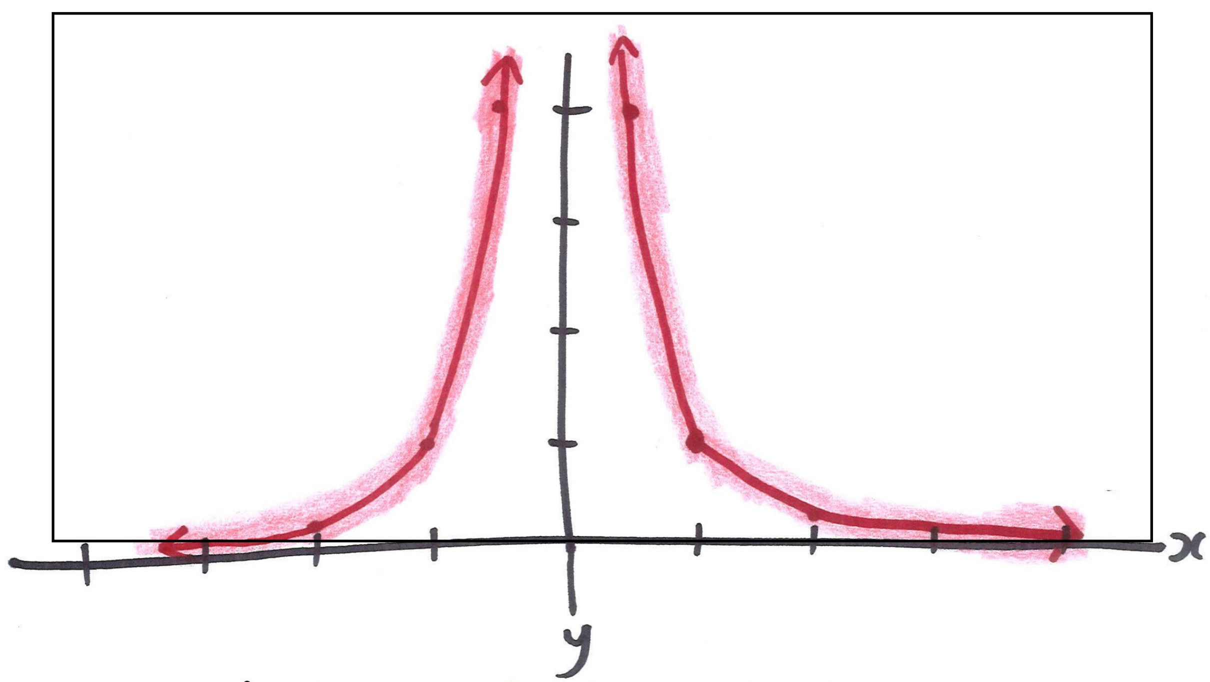
When x grew small,
y grew very big.

$$y = \frac{1}{x^2}$$

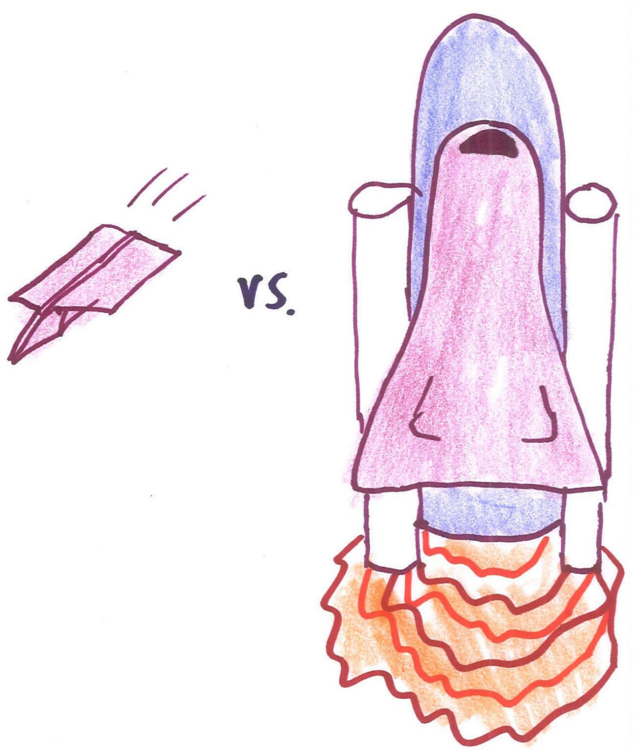


When x grew big,
y grew very, very small.

$$y = \frac{1}{x^2}$$



But no matter what,
y always stayed positive.



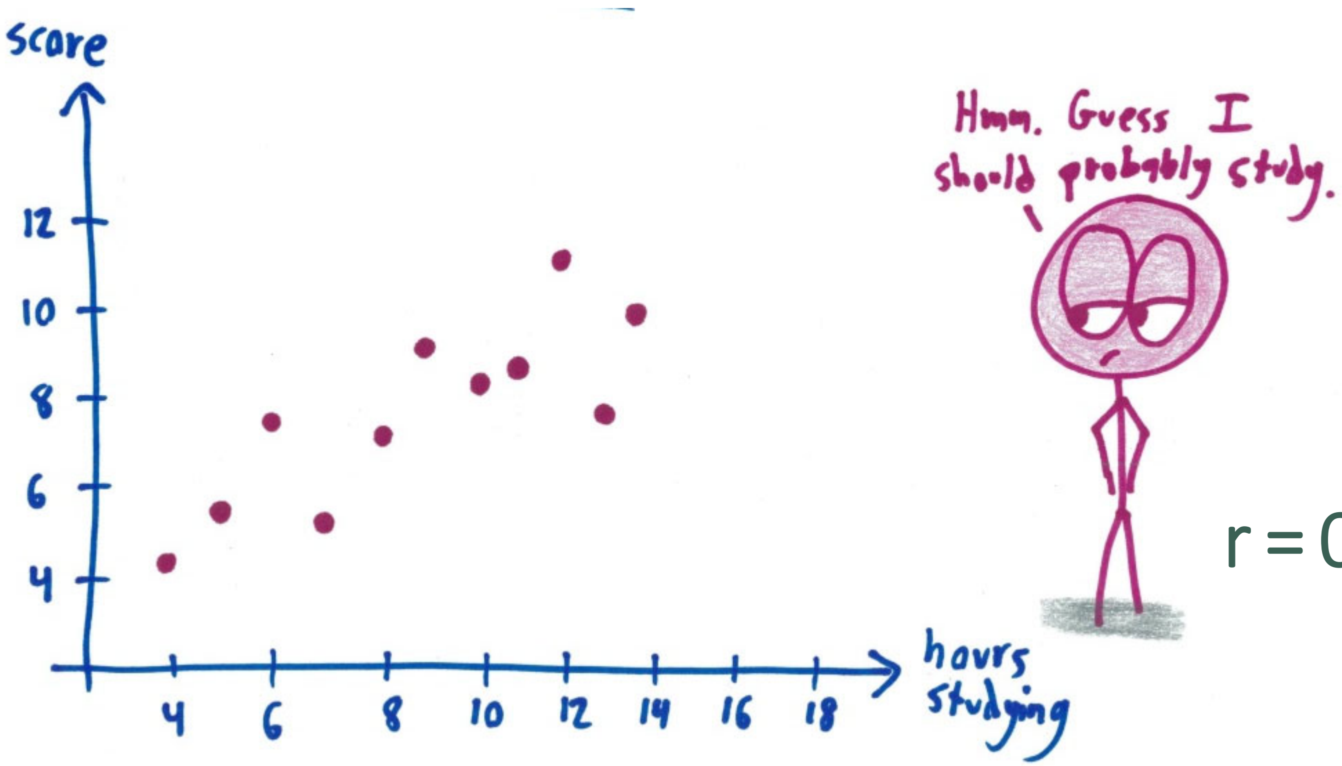
Expert readers **build narratives** out of mathematical notation.

Novices need help with this.

Example: Can we understand the formula for a correlation coefficient?

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

The **correlation coefficient** measures how closely two variables follow a linear relationship.



r = 0.816

x	*y*
10.0	8.04
8.0	6.95
13.0	7.58
9.0	8.81
11.0	8.33
14.0	9.96
6.0	7.24
4.0	4.26
12.0	10.84
7.0	4.82
5.0	5.68

What are the main "chunks" of this formula?

depends on x and y

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

only depends on x

only depends on y

What are the main “chunks” of this formula?

Covariance of X, Y

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

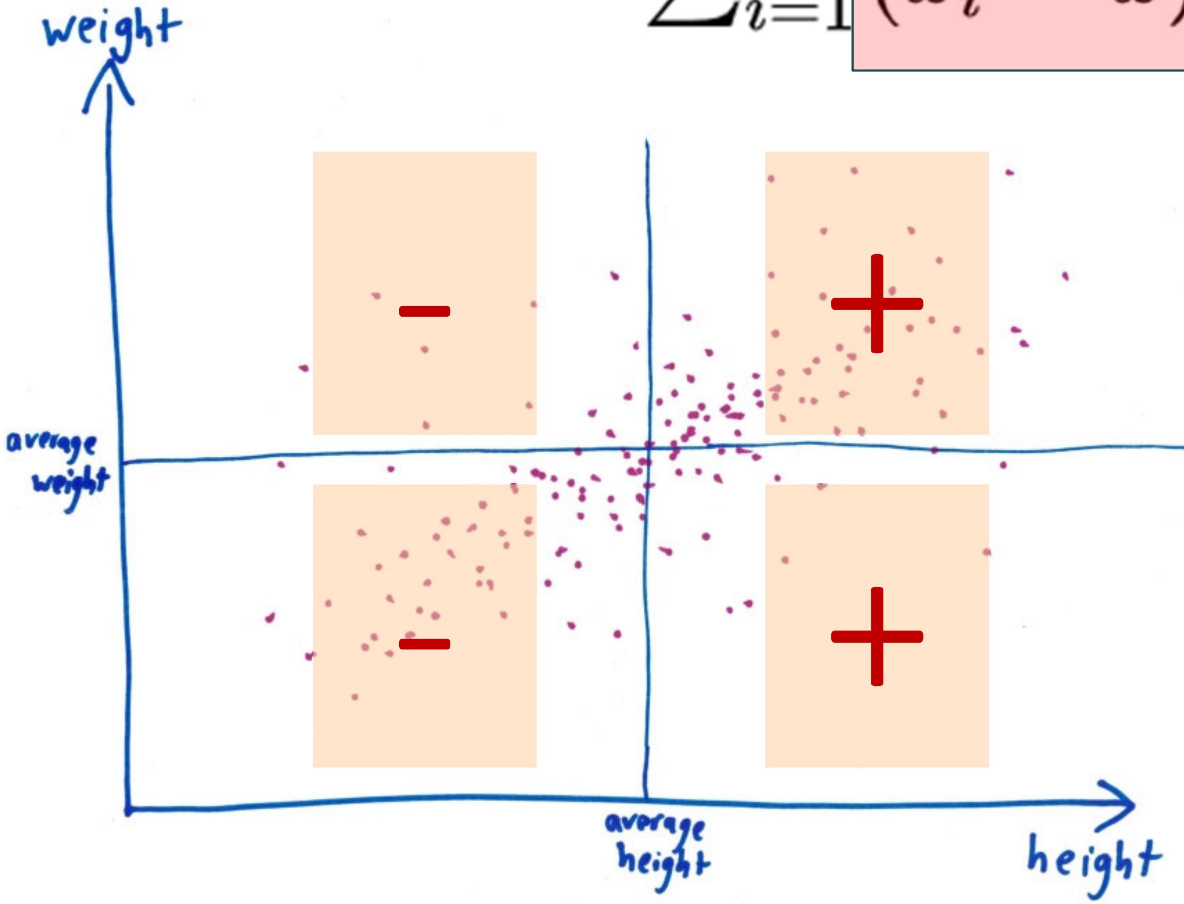
Variance of X

Variance of Y

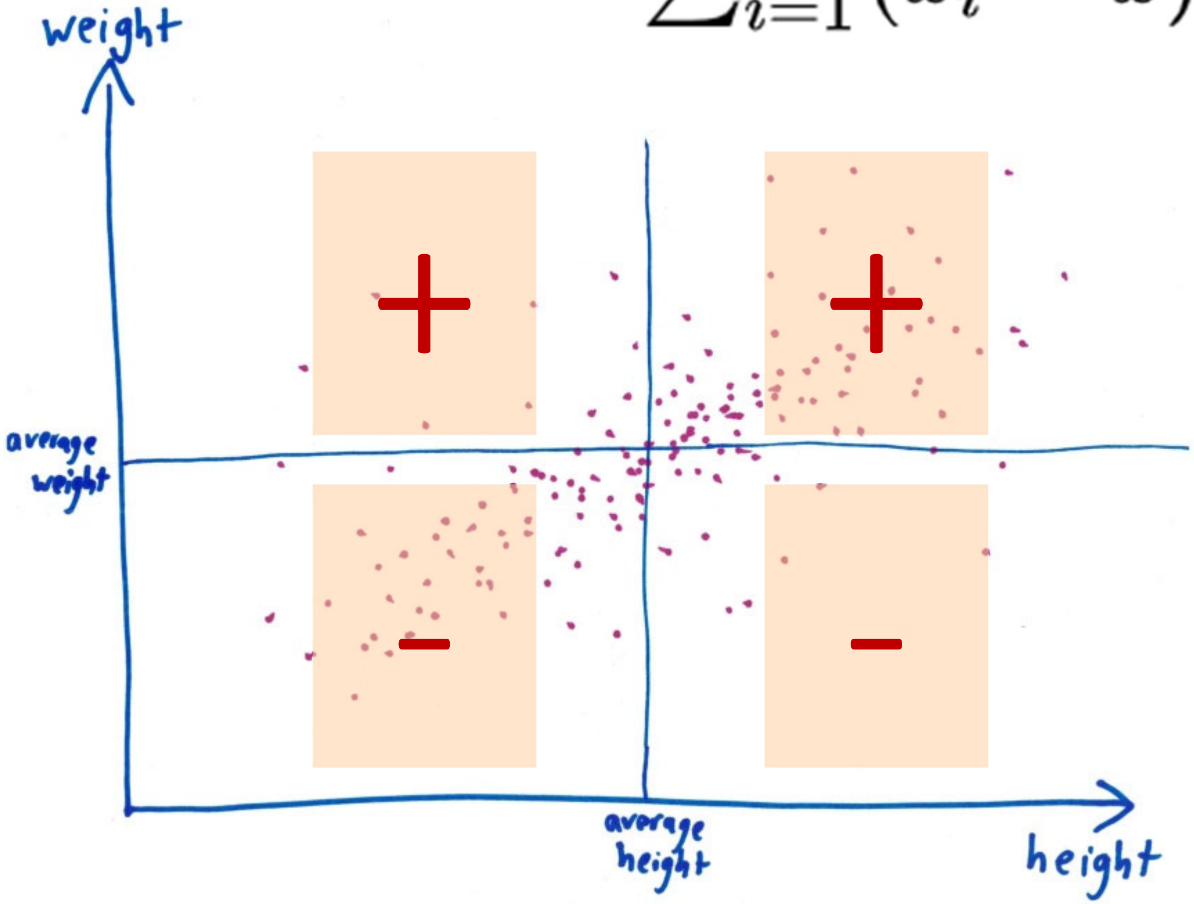
$$\rho_{X,Y} = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y}$$

Now, can an area model help us?

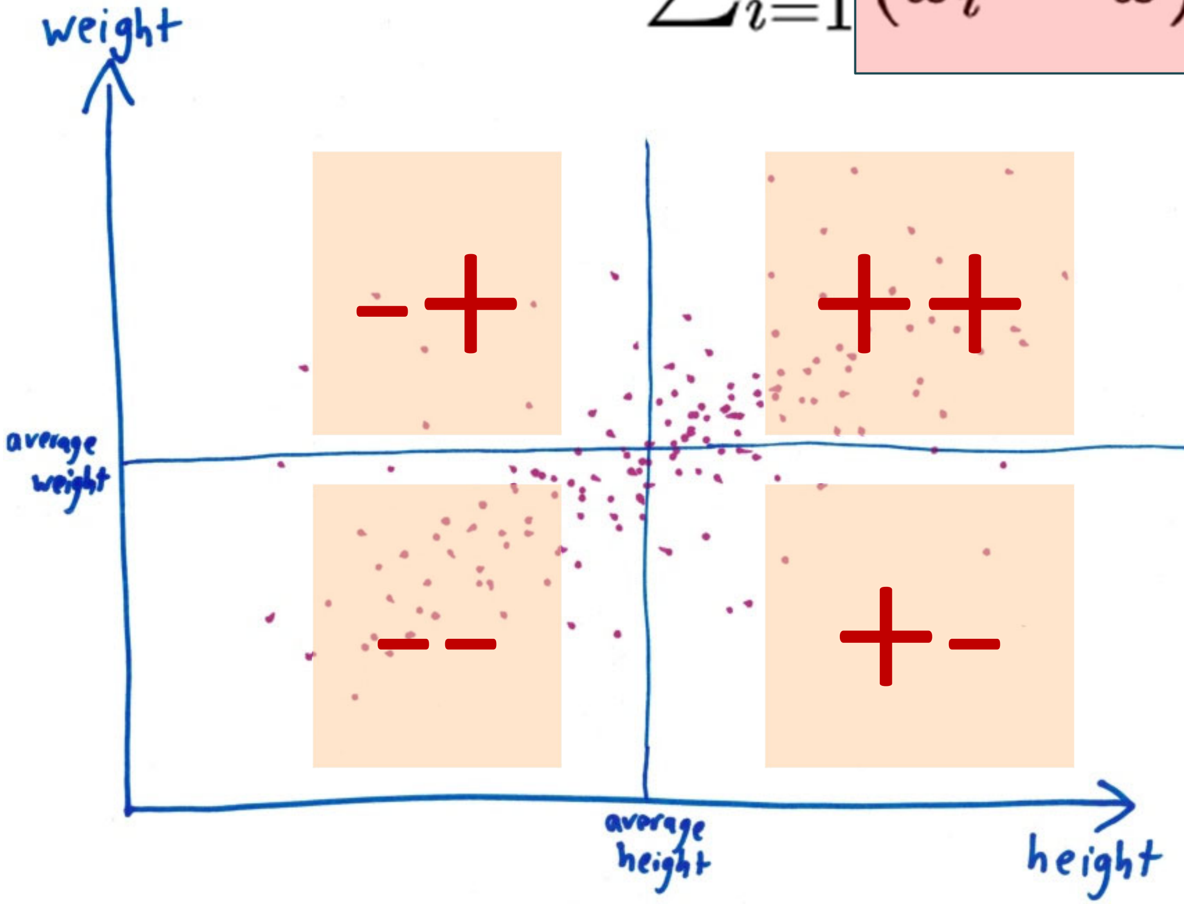
$$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$



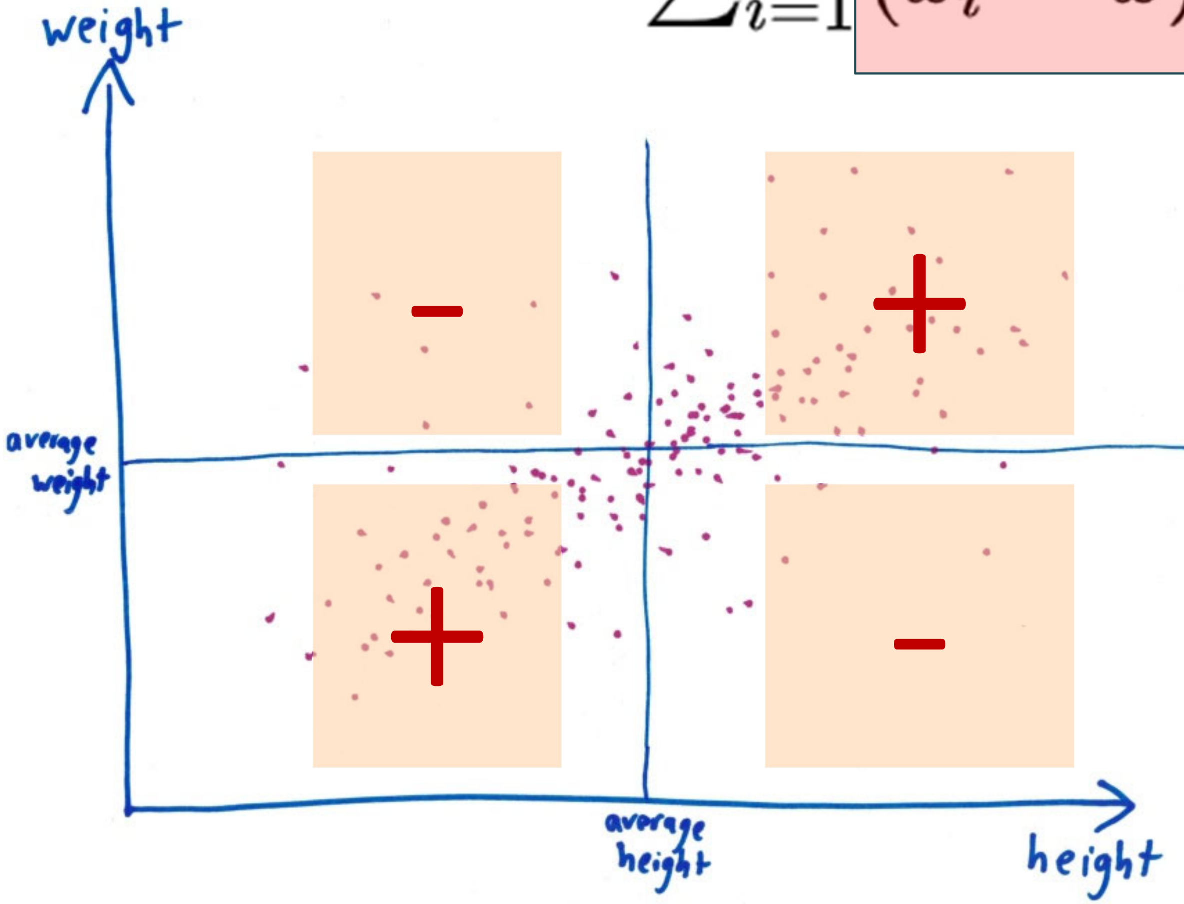
$$\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$



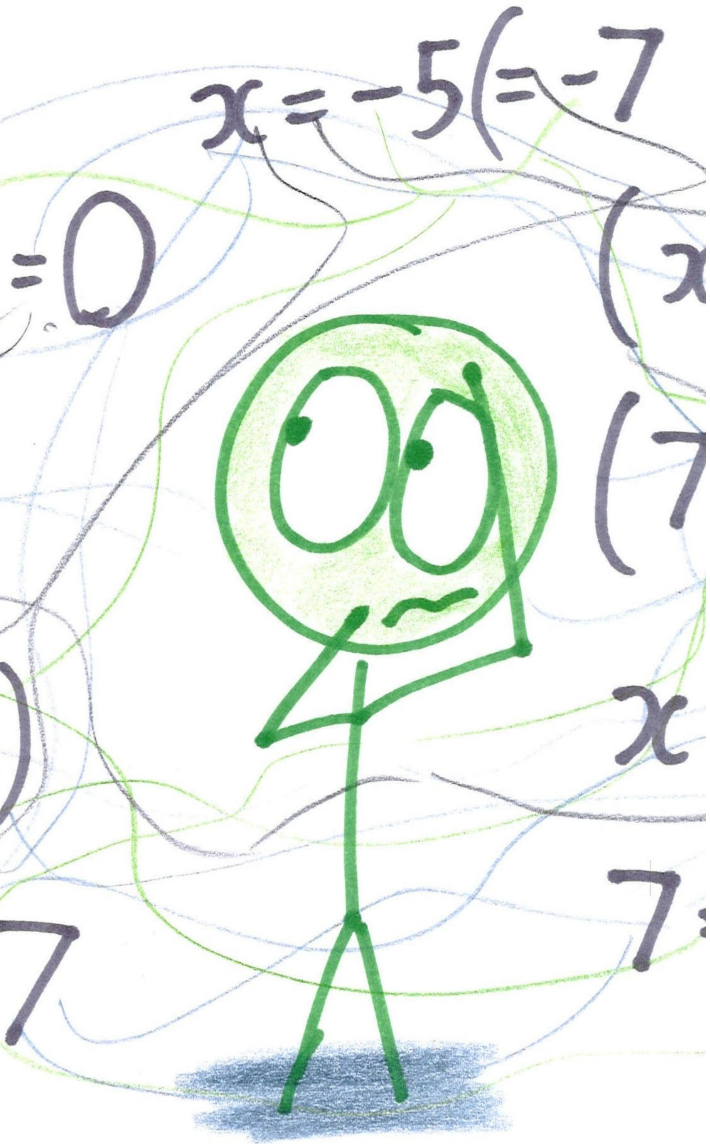
$$\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$



$$\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$



Now, can we tell a narrative?



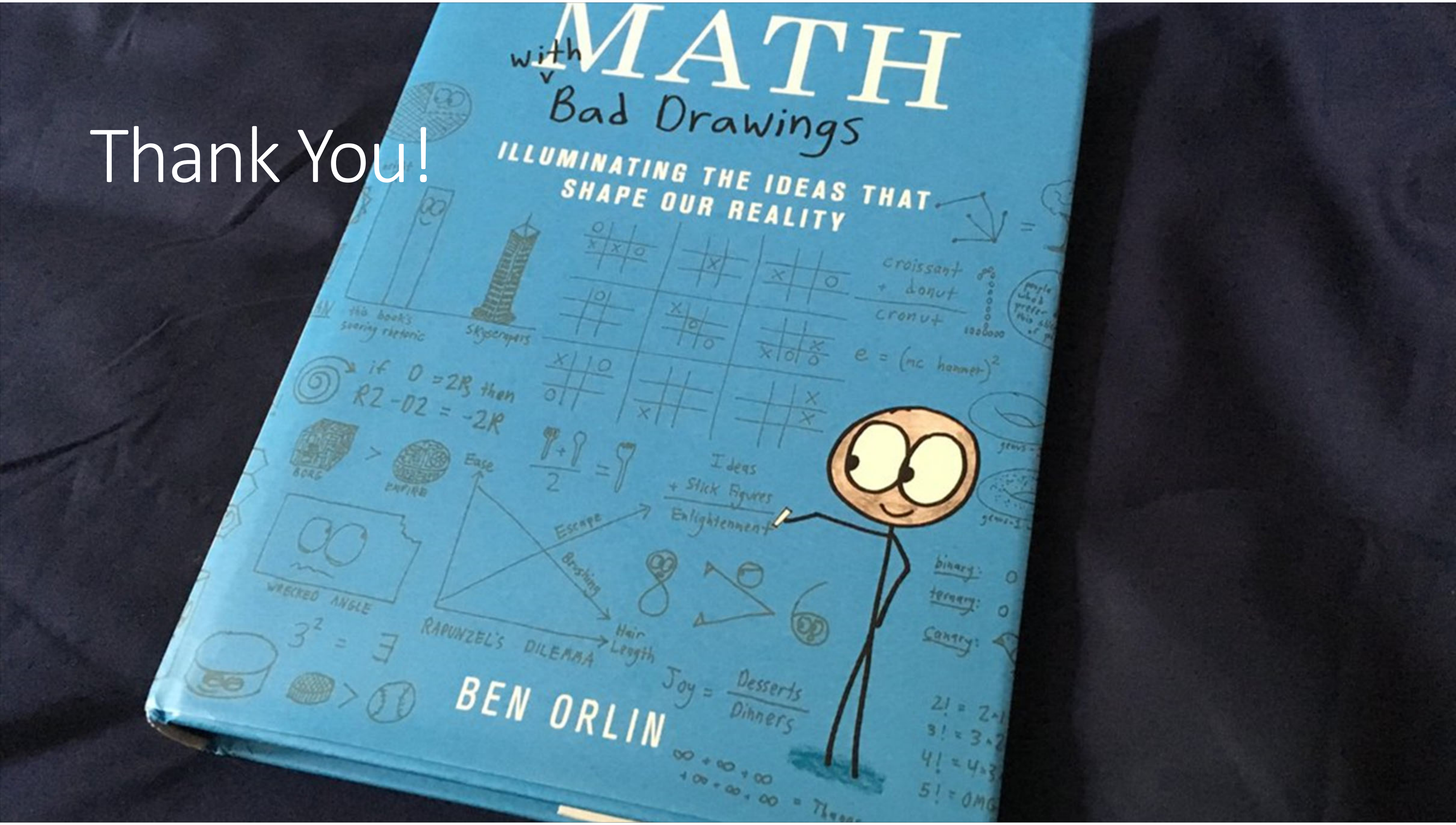
What were our strategies?

1. **Chunking** a big expression into a few major pieces.

2. **Drawing area models** for acts of multiplication.

3. **Telling narratives** about how the variables relate.

Thank You!


with MATH
Bad Drawings
ILLUMINATING THE IDEAS THAT SHAPE OUR REALITY
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