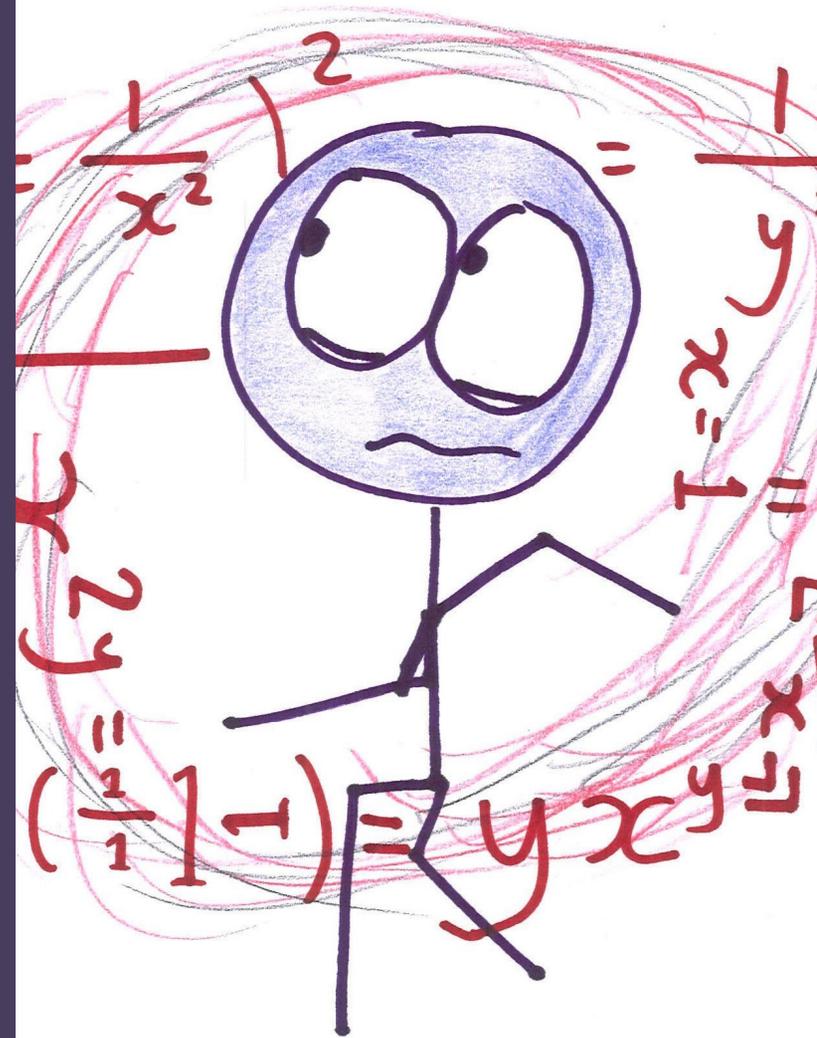


What Does Algebra Look Like to Students?

Ben Orlin

Math with Bad Drawings

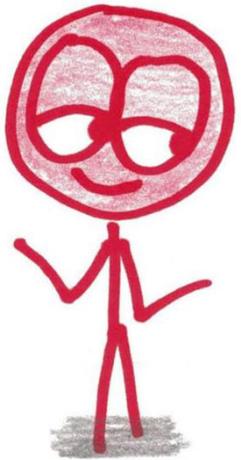


Example: Can we understand the formula for a correlation coefficient?

$$r_{xy} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}$$

$$r_{xy} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}$$

"Ah yes, of course."

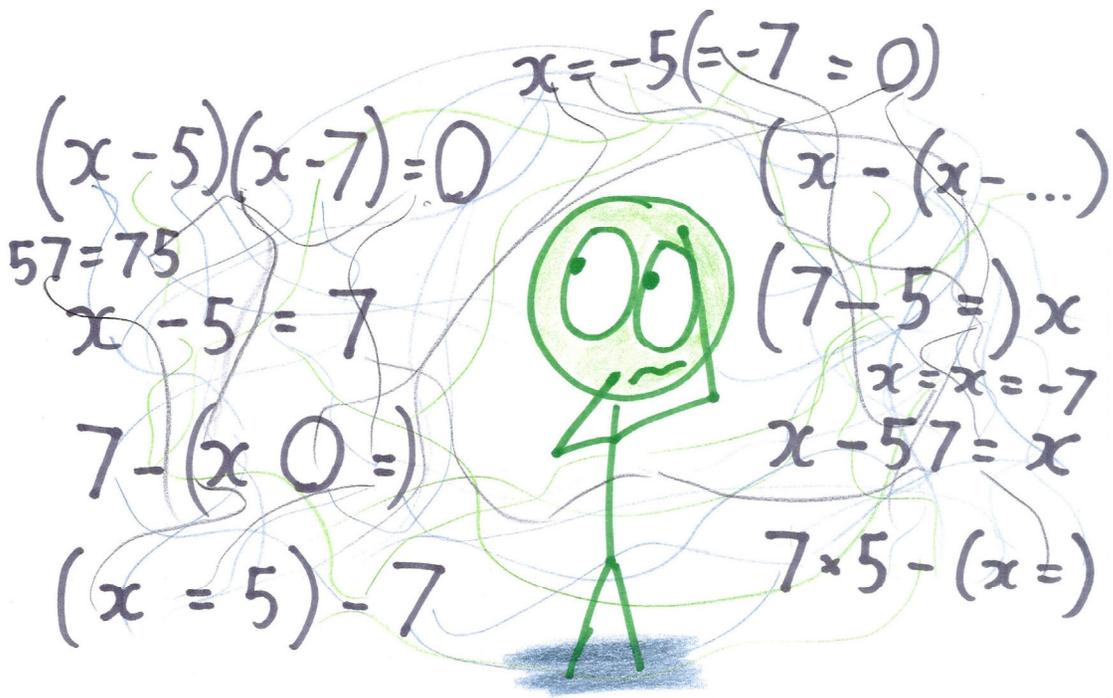


"Um... yeah, looks right."

"Kinda losing me, bro."



"PLEASE. NO."



Algebraic notation is language.

Experts have reading strategies.

Novices may not.

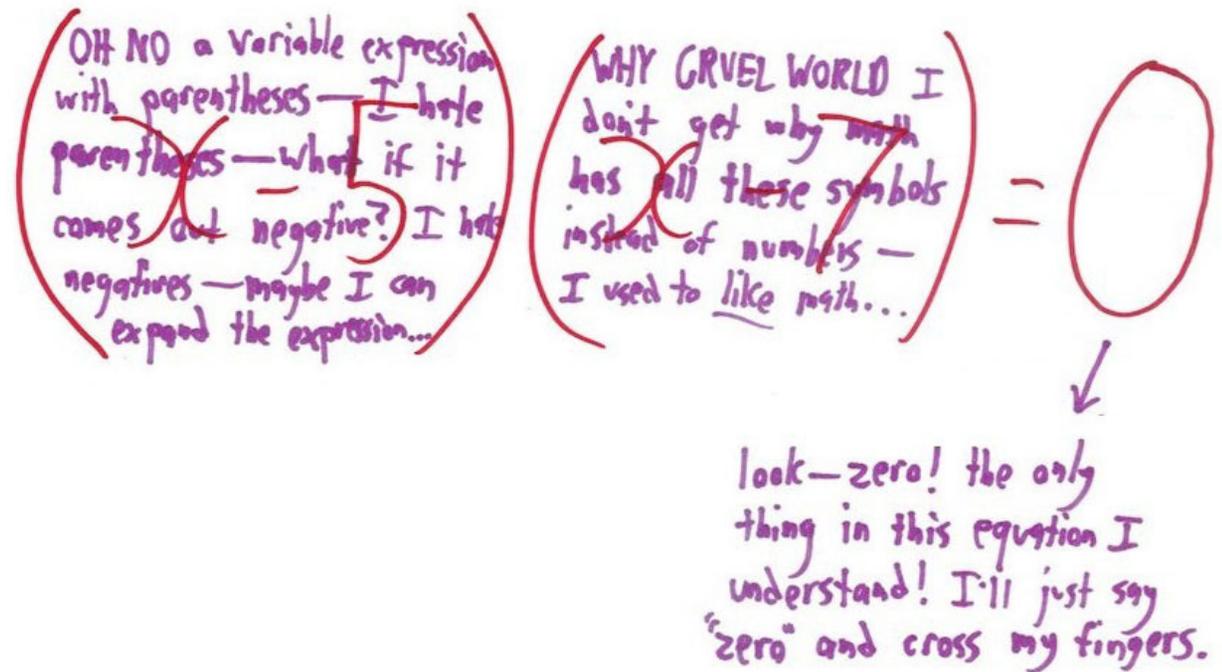
A challenge for teachers: Identify and unpack these reading strategies.

What do
novices and
experts see?

$$(x-5)(x-7)=0$$

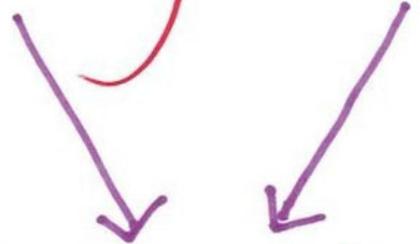
Solve for x .

What A Novice Sees



What An Expert Sees

Something \times Something = 0



ah - so one of these shady characters must be zero.

What do
novices and
experts see?

$$y = 7 - (x + 5)^2$$

Find the graph's vertex.

What A Novice Sees

The image shows handwritten mathematical work in blue and red ink. At the top, the equation $h(y) = 7 = (x + 5)^2$ is written. Below this, a coordinate system is drawn with a vertical y-axis and a horizontal x-axis. The y-axis is labeled $y = 7$ at the top. The x-axis is labeled $x = -5$ at the left. The origin is marked with a question mark and the word "minimum?". To the right of the origin, the number 5 is written, with a note below it that says "not no, -5". Arrows point from the 7 in the equation to the $y = 7$ label, and from the $(x + 5)^2$ part of the equation to the $x = -5$ label. The entire diagram is heavily scribbled over with multiple overlapping lines in blue and red.

What An Expert Sees

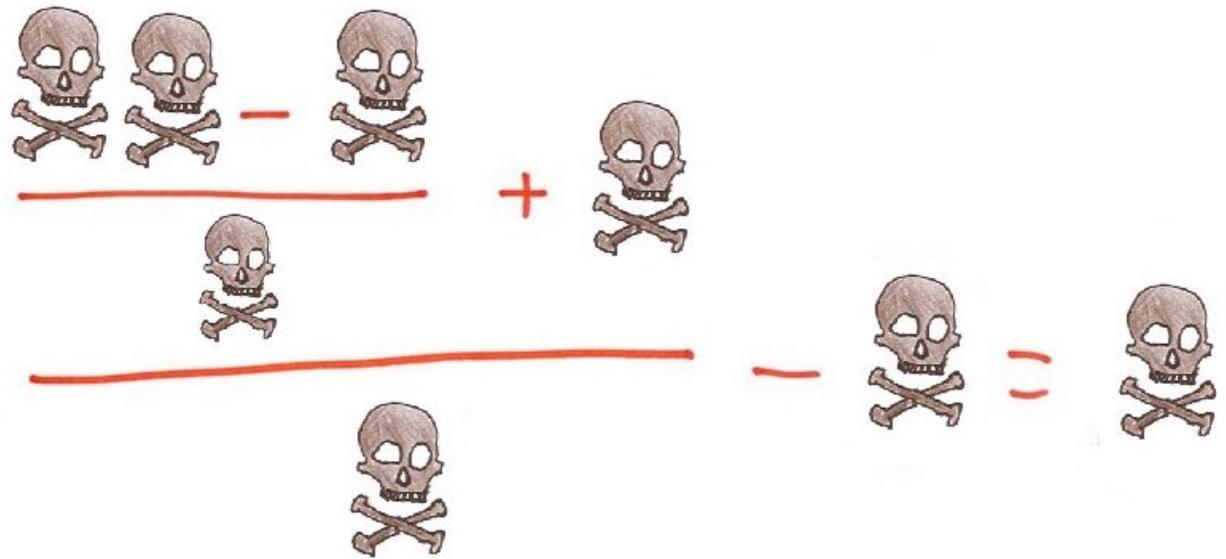
$$y = 7 - (\text{some positive number})$$

so, maximized at $y=7$,
when the thing in brackets is 0

What do
novices and
experts see?

$$\frac{\frac{7x-1}{2} + 4}{7} - 3 = 8$$

What a Novice Sees



What an Expert Sees

Something

↓
eh, I'll worry about
this bit later

$$-3 = 8$$

$$\text{Something} \times \text{Something} = 0$$

ah—so one of these shabby characters must be zero.

$$y = 7 - (\text{some positive number})$$

so, maximized at $y=7$,
when the thing in brackets is 0

$$\boxed{\text{Something}} - 3 = 8$$

eh, I'll worry about this bit later

Each of these is an example of what a psychologist would call chunking.

You break a large whole into a few meaningful pieces.

$$7 \times 11 \times 13$$

What do
novices and
experts see?

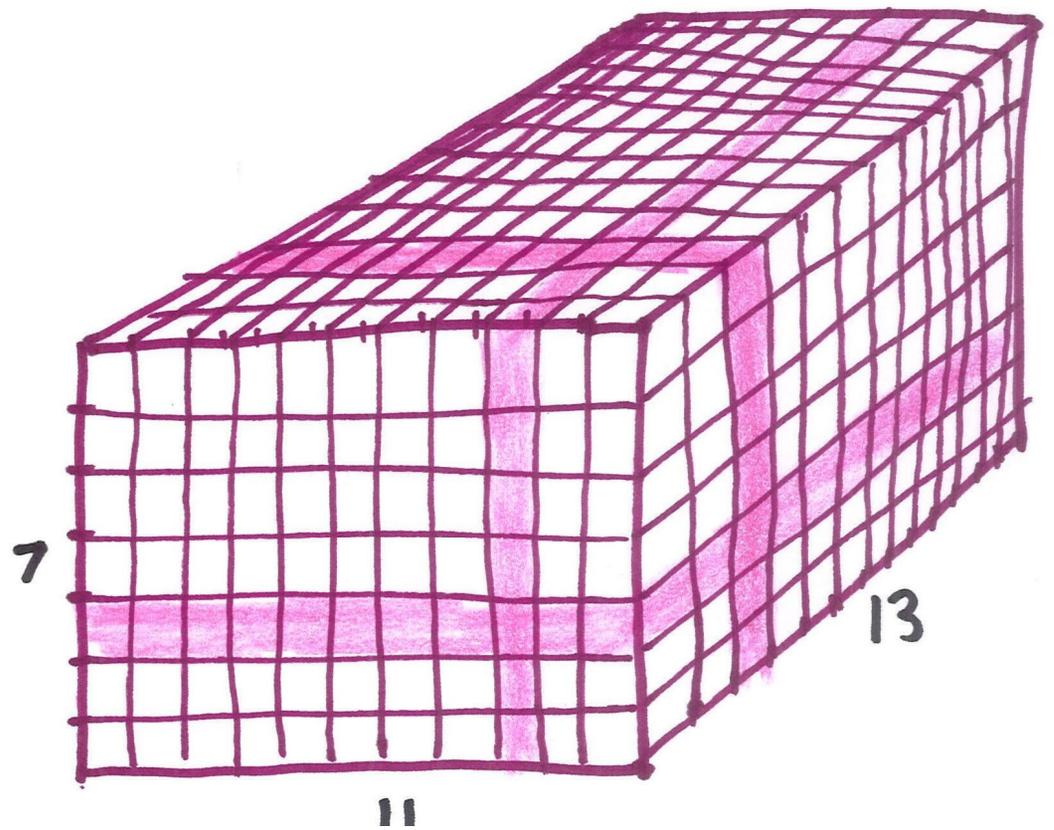
What a
novice
sees

$$7 \times 11 \times 13$$



$$7 \times 11 \times 13$$

What an
expert
sees



What do
novices and
experts see?

$$A = \pi r^2$$

What a Novice Sees

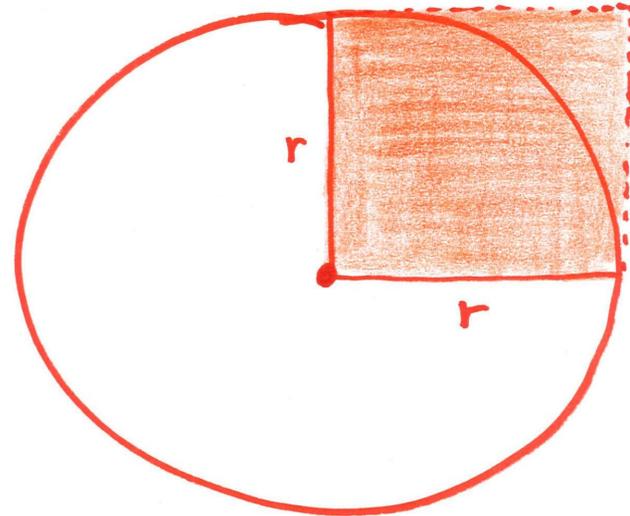
$$A = \pi r^2$$

Area equals pi r squared.
Area equals pi r squared.



What an Expert Sees

$$A = \pi r^2$$



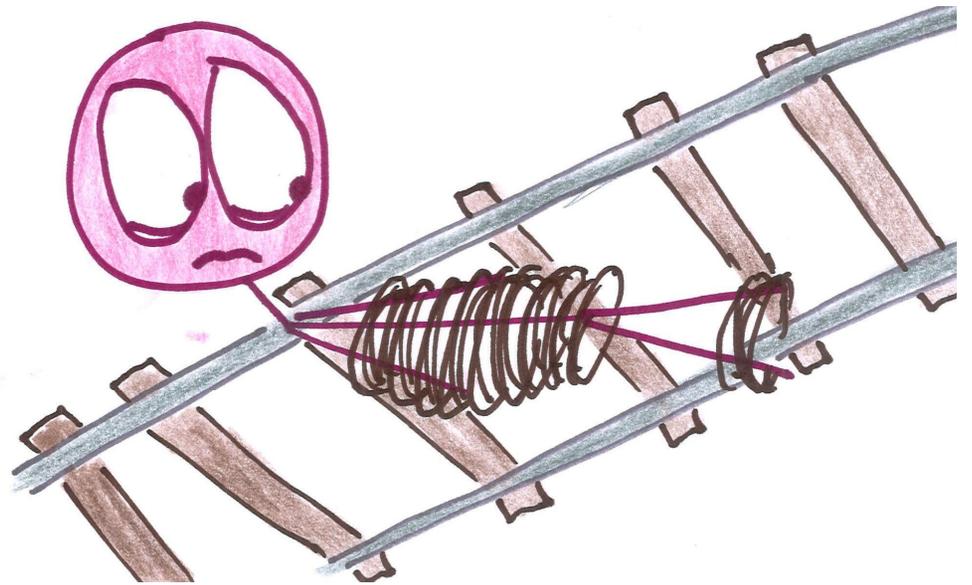
"To fill the circle, you'd need π squares."
a little \swarrow more than 3

What do
novices and
experts see?

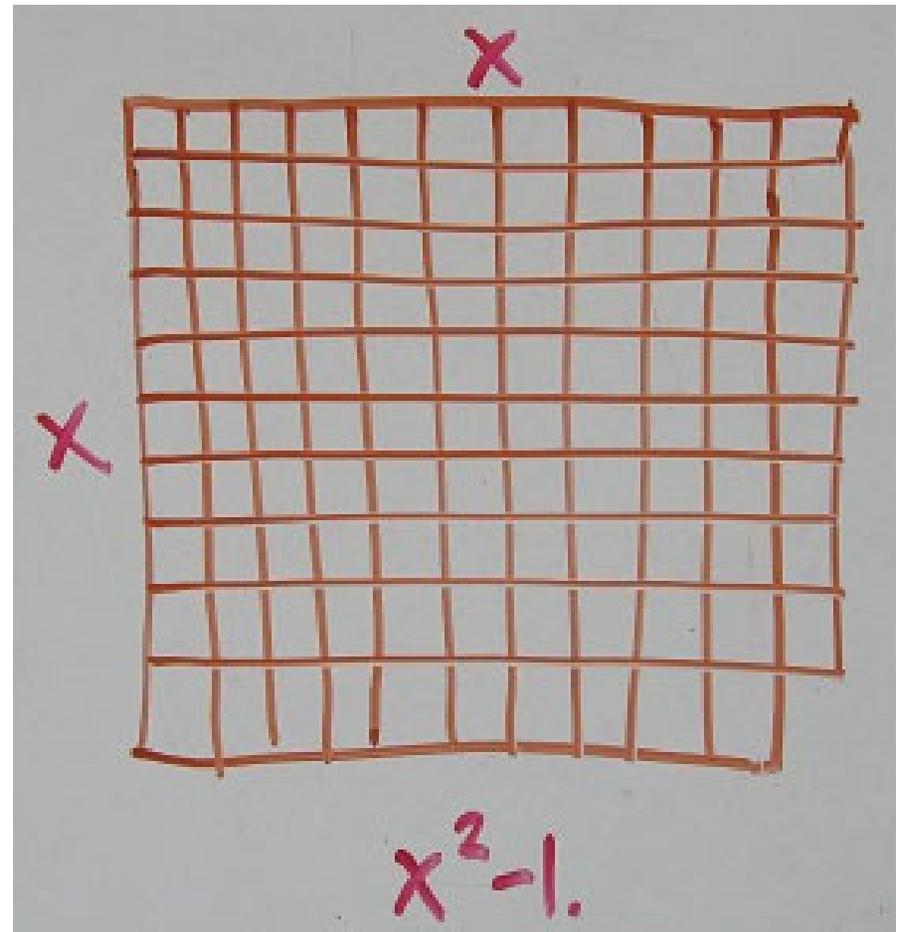
$$x^2 - 1 = (x - 1)(x + 1)$$

$$x^2 - 1 = (x - 1)(x + 1)$$

What a
novice
sees

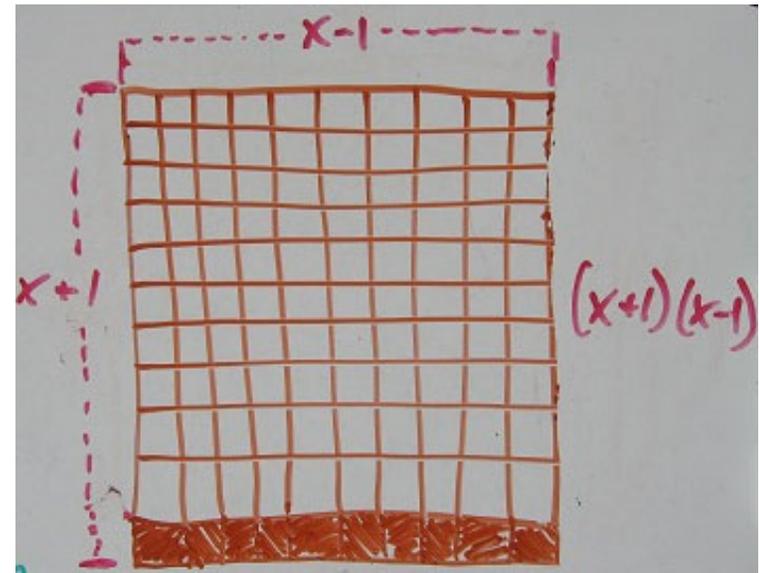
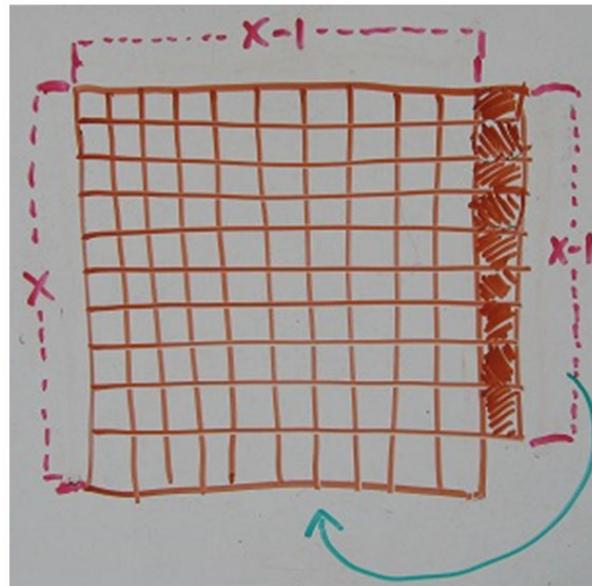
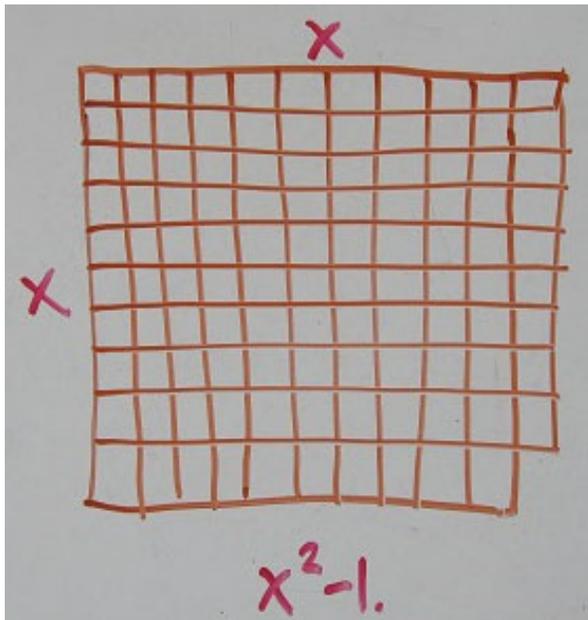


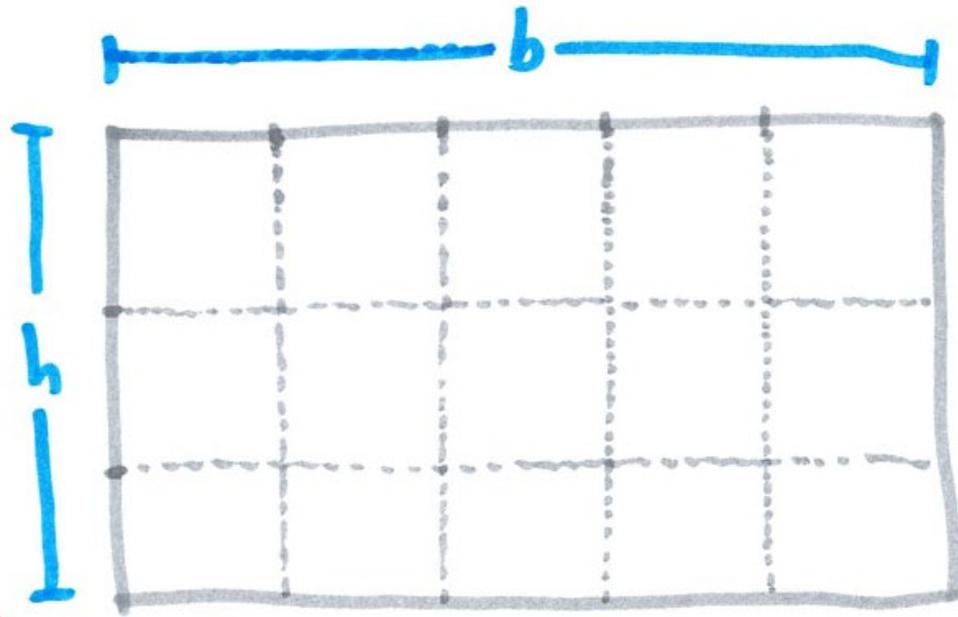
$$x^2 - 1 = (x - 1)(x + 1)$$



What an
Expert
Sees

$$x^2 - 1 = (x - 1)(x + 1)$$





$$A = bh$$

I find area models really powerful for making sense of algebra.

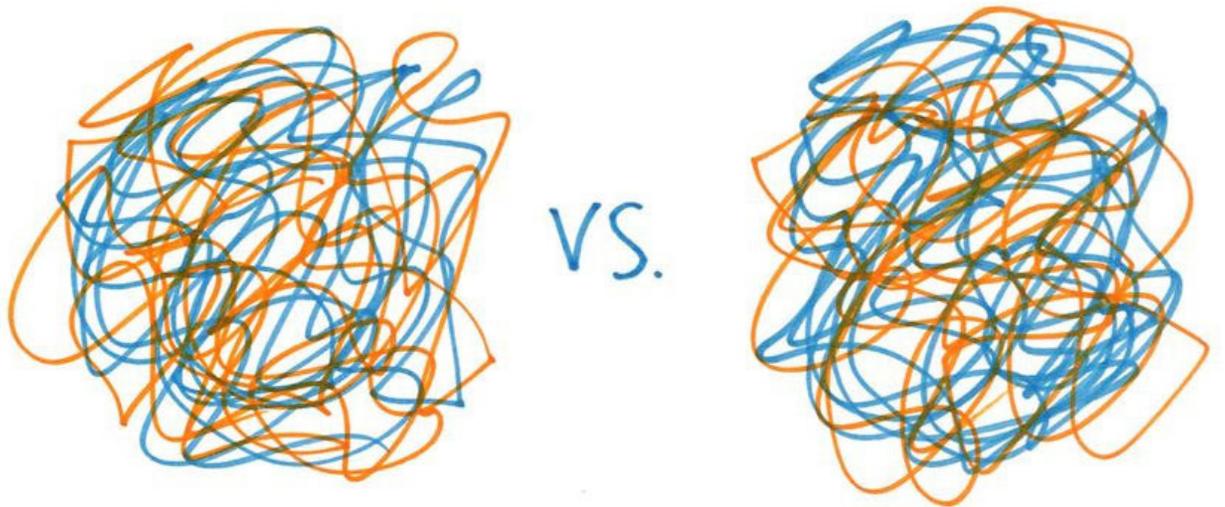
But I struggle to get my students on board – at least, at first.

What do
novices and
experts see?

$$x^2 \quad \text{vs.} \quad 2^x$$

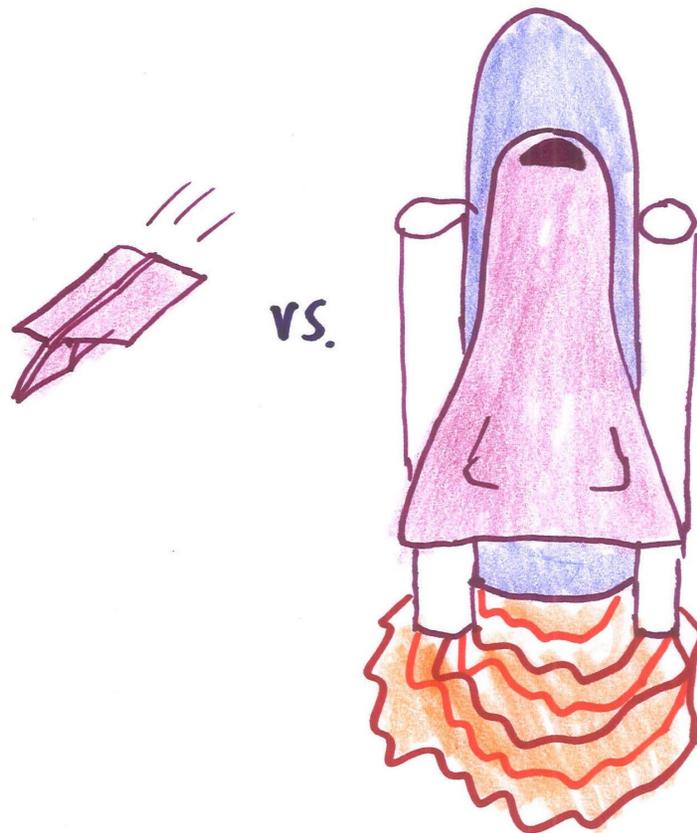
What a
Novice Sees

$$x^2 \text{ vs. } 2^x$$

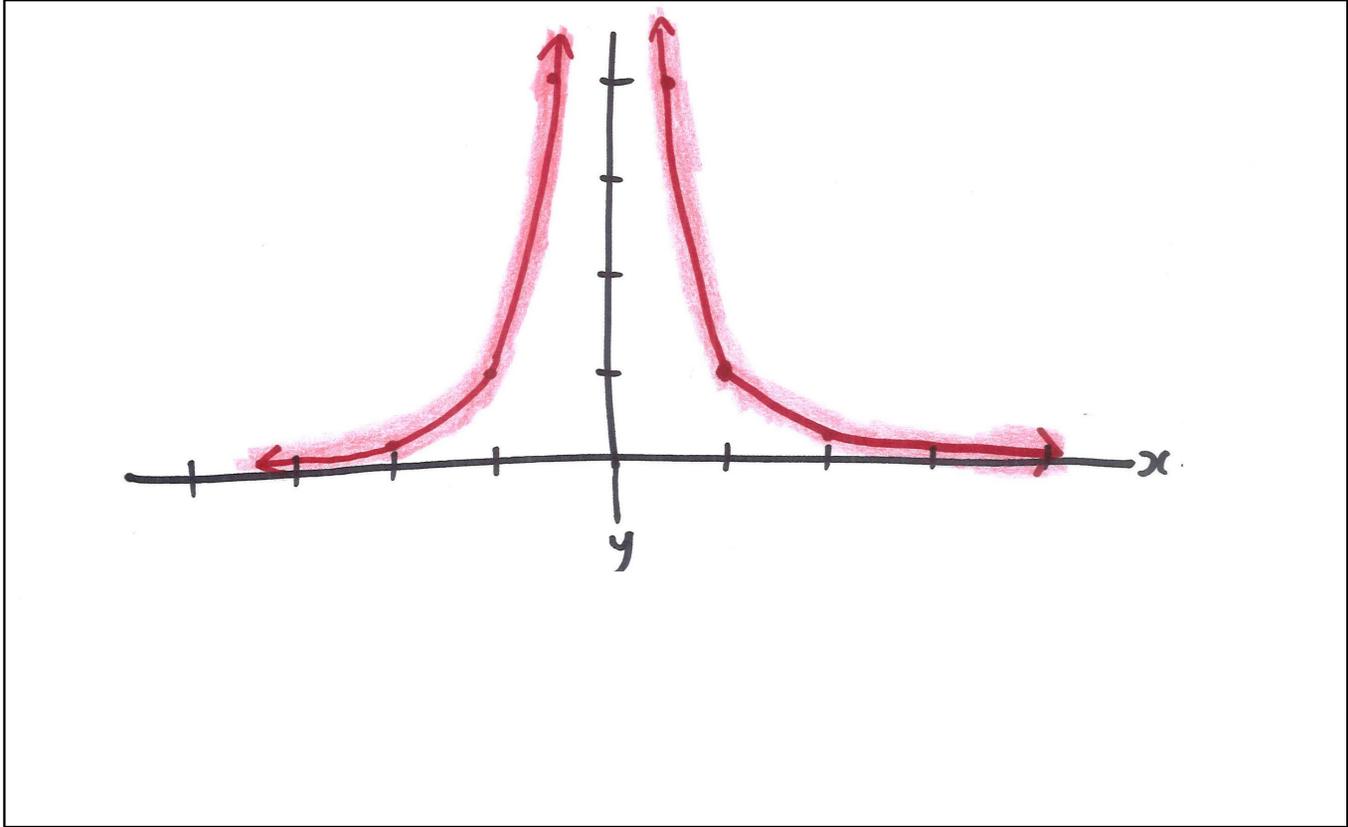


What an
Expert Sees

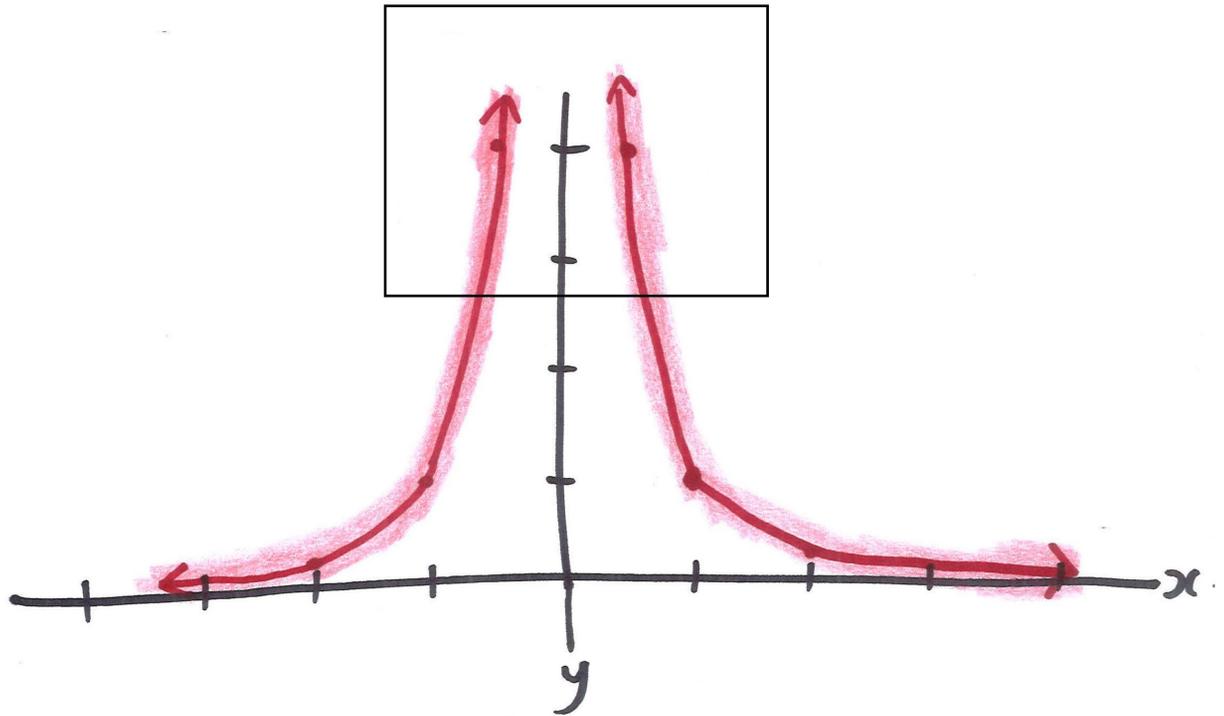
x^2 vs. 2^x



$$y = \frac{1}{x^2}$$

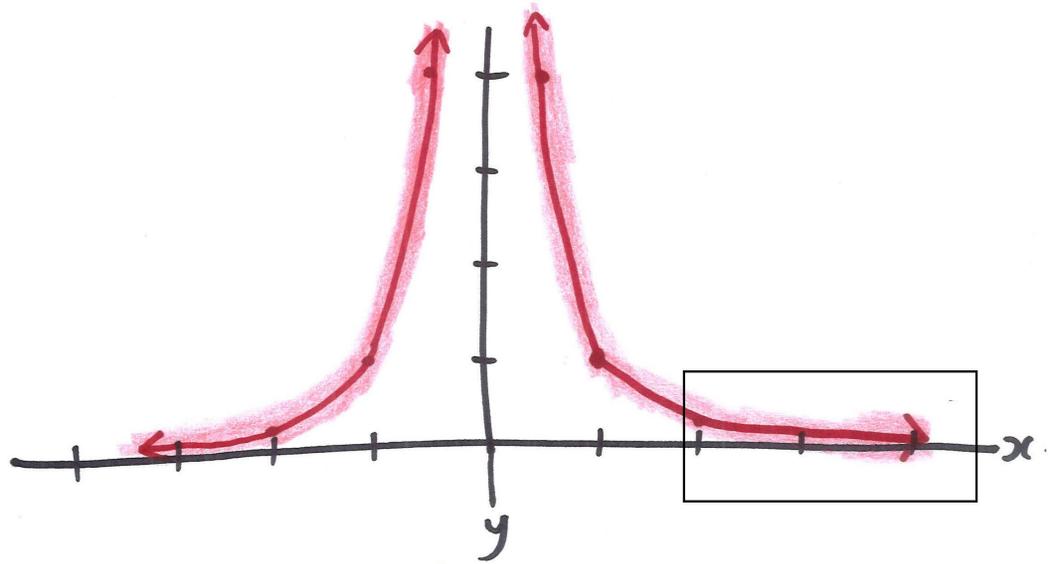


$$y = \frac{1}{x^2}$$



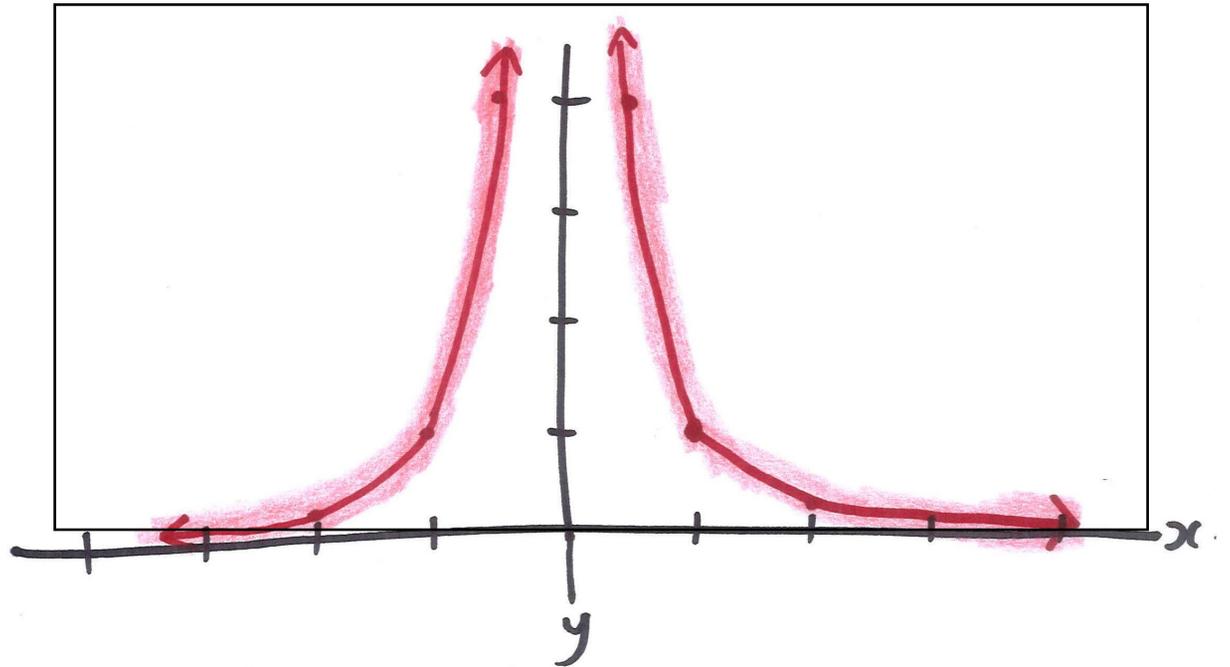
When x grew small,
 y grew very big.

$$y = \frac{1}{x^2}$$

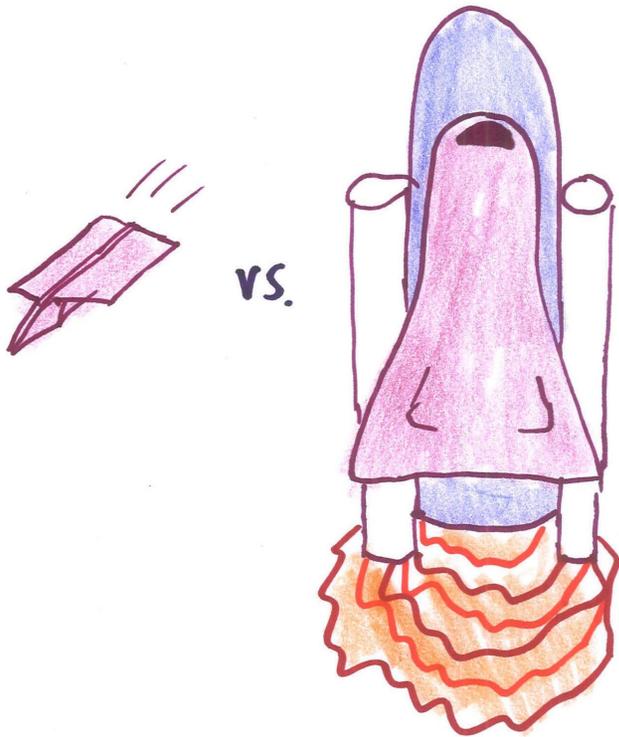


When x grew big,
 y grew very, very small.

$$y = \frac{1}{x^2}$$



But no matter what,
y always stayed positive.



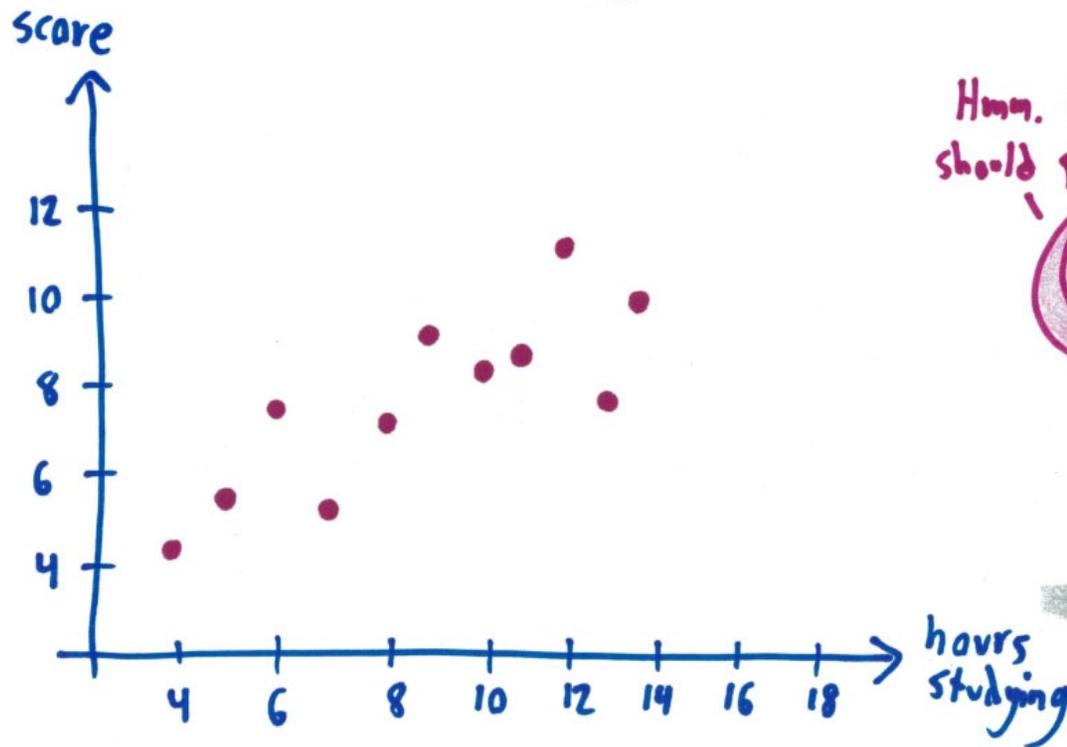
Expert readers
build narratives out
of mathematical
notation.

Novices need help
with this.

Example: Can we understand the formula for a correlation coefficient?

$$r_{xy} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}$$

The correlation coefficient measures how closely two variables follow a linear relationship.



$$r = 0.816$$

x	y
10.0	8.04
8.0	6.95
13.0	7.58
9.0	8.81
11.0	8.33
14.0	9.96
6.0	7.24
4.0	4.26
12.0	10.84
7.0	4.82
5.0	5.68

What are the main “chunks” of this formula?

depends on x and y

$$\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

$r_{xy} =$

$$\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}$$

only depends on x

$$\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}$$

only depends on y

What are the main “chunks” of this formula?

Covariance of X, Y

$$\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

$r_{xy} =$

$$\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}$$

Variance of X

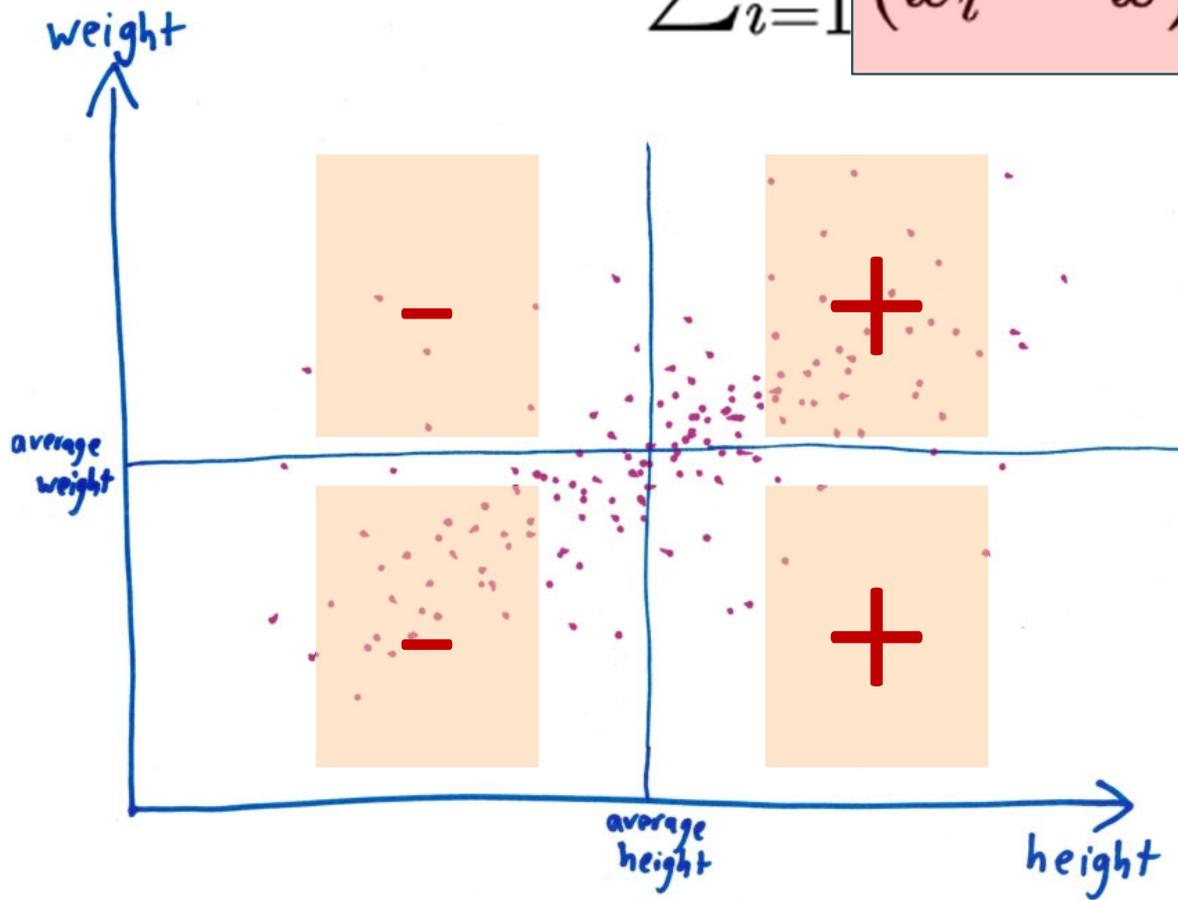
$$\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}$$

Variance of Y

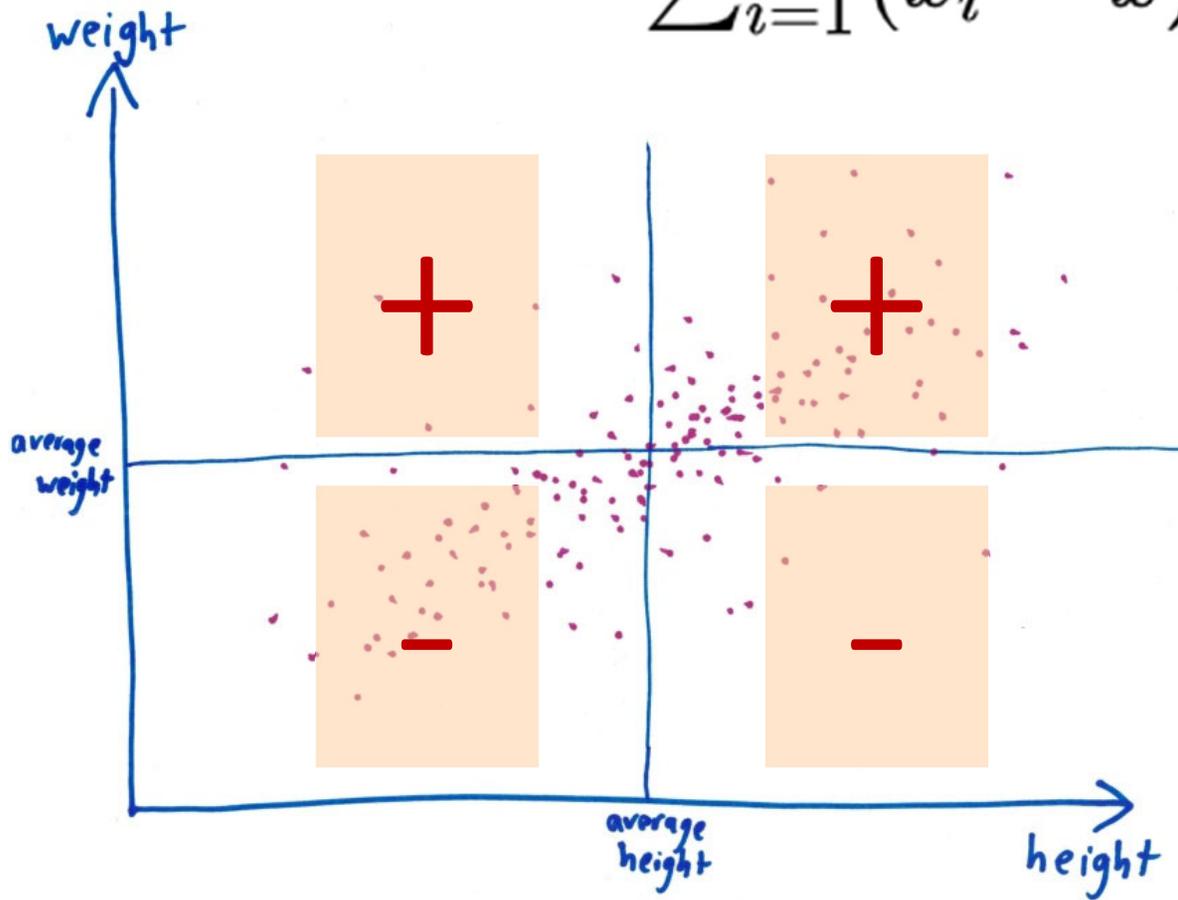
$$\rho_{X,Y} = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$$

Now, can an area model help us?

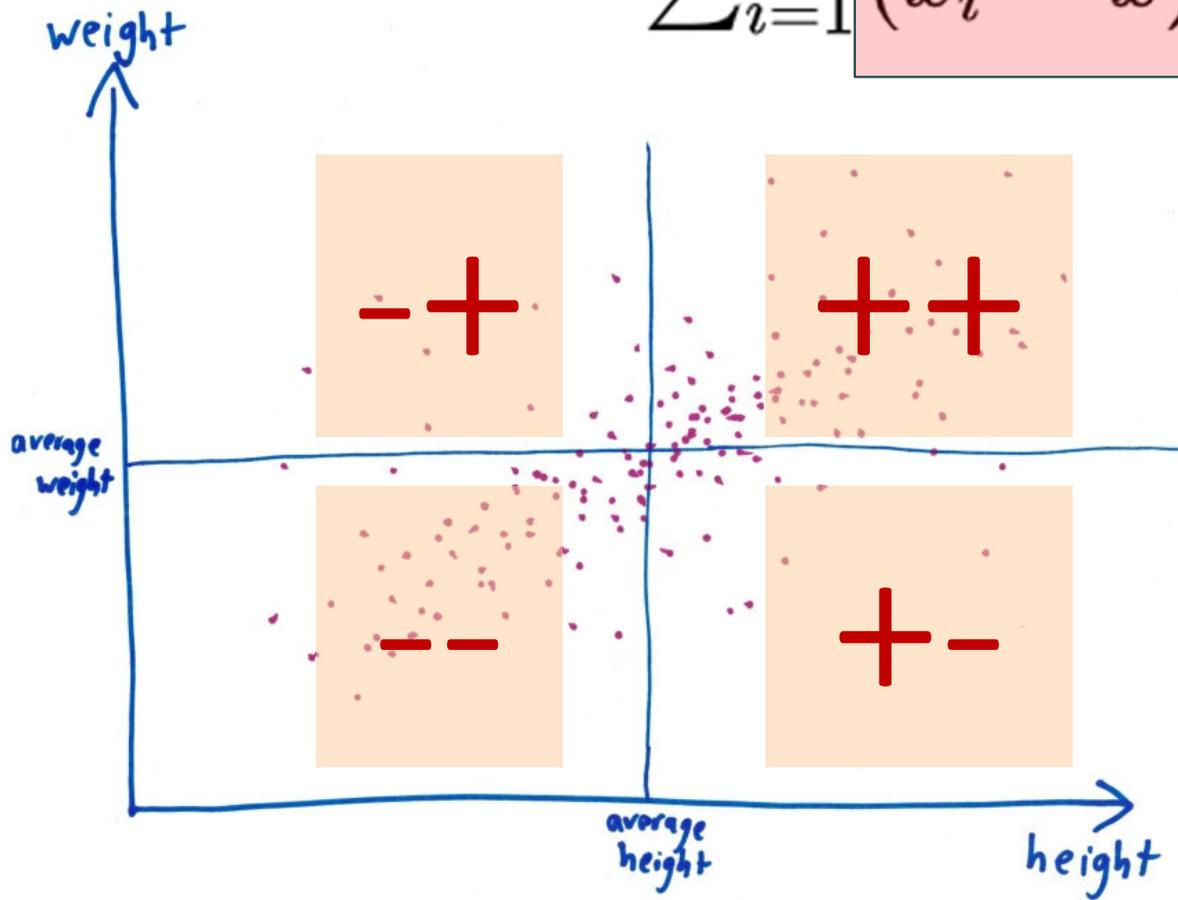
$$\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$



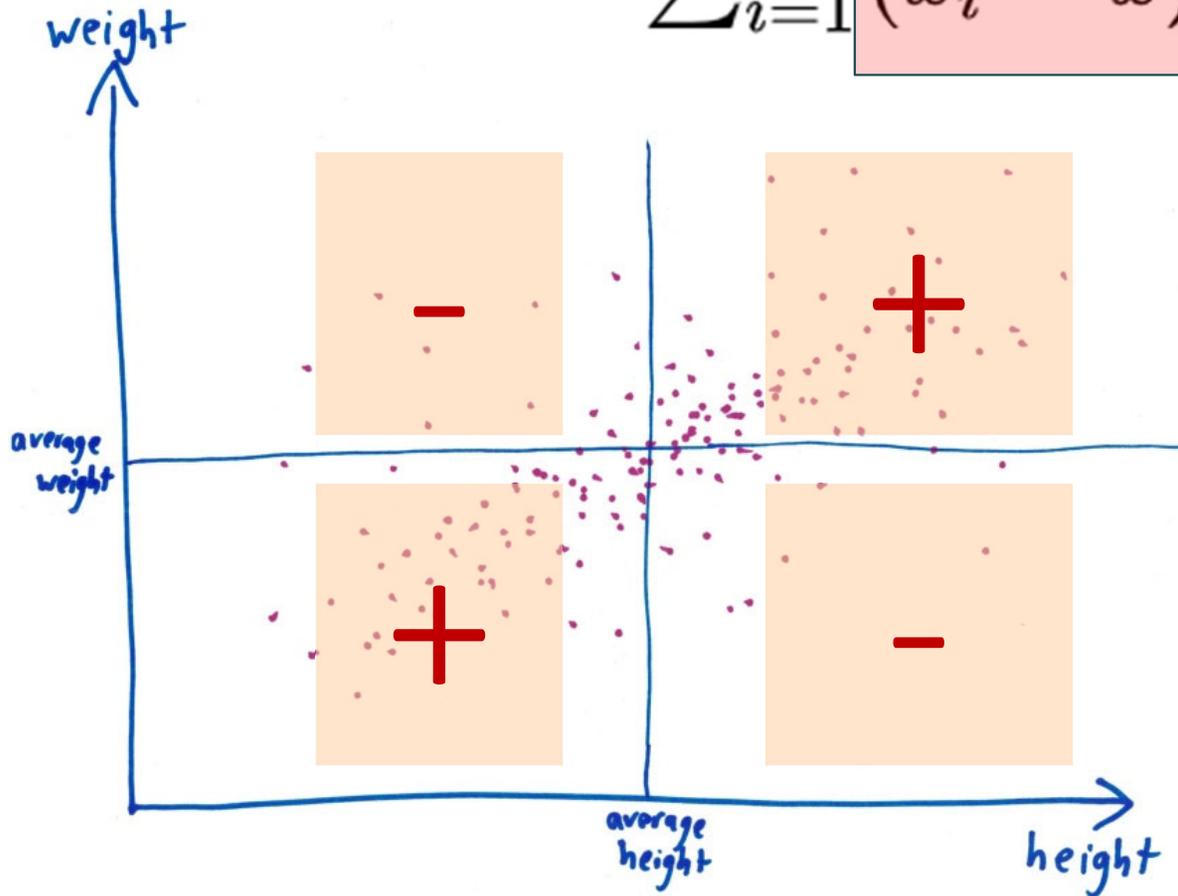
$$\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$



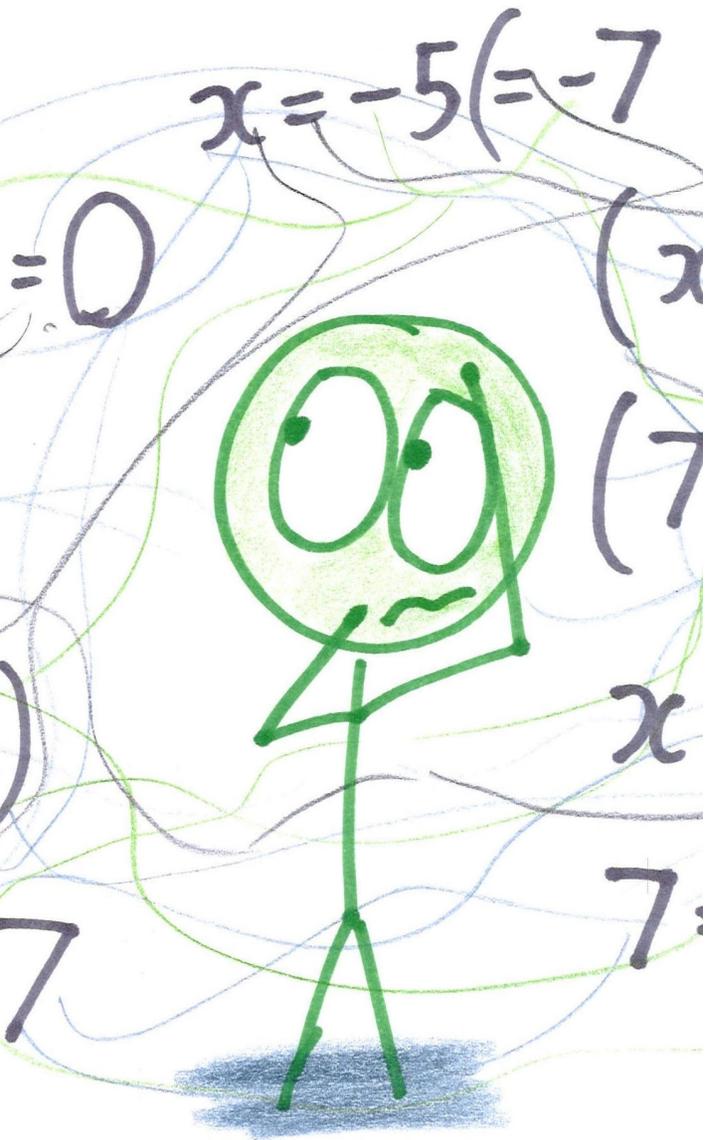
$$\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$



$$\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$



Now, can we tell a narrative?



What were our strategies?

1. Chunking a big expression into a few major pieces.
2. Drawing area models for acts of multiplication.
3. Telling narratives about how the variables relate.

Thank You!

